Automatic Control Theory

CSE 322

Lec. 4
Mathematical Modeling of Dynamic System

2. Mechanical Systems

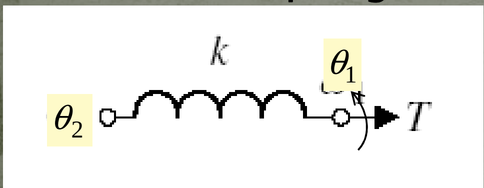
Mechanical Systems

- **Part-I:** Translational Mechanical System
- **Part-II:** Rotational Mechanical System
- **Part-III:** Mechanical Linkages

2. Rotational Mechanical Systems

Basic Elements of Rotational Mechanical Systems

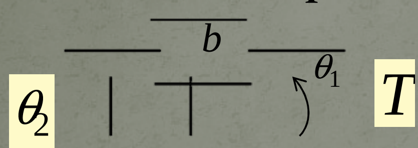
Rotational Spring



$$T = k(\theta_1 - \theta_2)$$

Basic Elements of Rotational Mechanical Systems

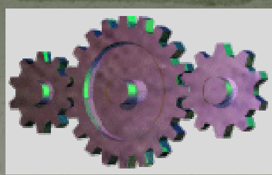
Rotational Damper



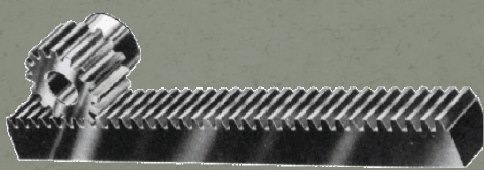
$$T = b(\dot{\theta}_1 - \dot{\theta}_2)$$

3. Mechanical Linkages

Gear

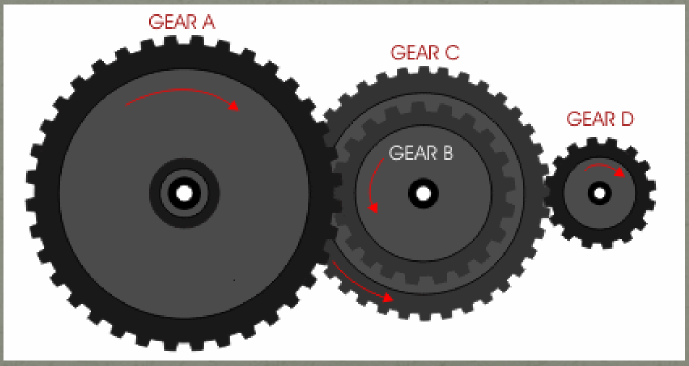


- **Gear** is a toothed machine part, such as a wheel or cylinder, that meshes with another toothed part to transmit motion or to change speed or direction.



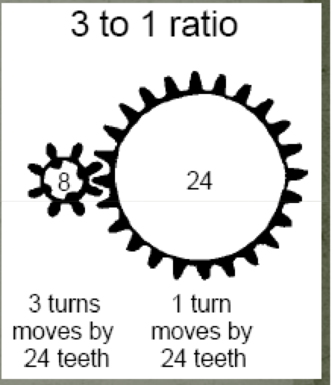
Fundamental Properties

- The two gears **turn in opposite** directions: one clockwise and the other counterclockwise.
- Two gears revolve at different speeds when number of teeth on each gear are **different**.



Gearing Up and Down

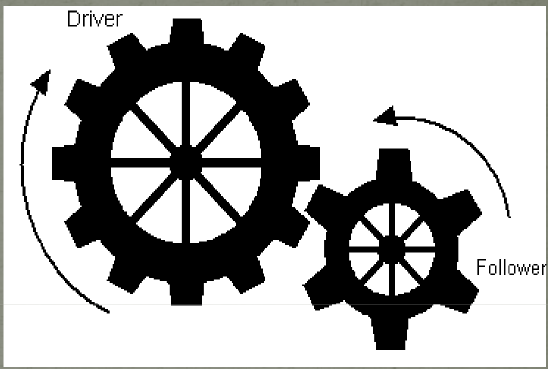
- **Gearing up** is able to convert torque to velocity.
- The more velocity gained, the more torque sacrifice.
- The ratio is exactly the same: if you get three times your original angular velocity, you reduce the resulting torque to one third.
- This conversion is symmetric: we can also convert velocity to torque at the same ratio.
- The price of the conversion is power loss due to friction.

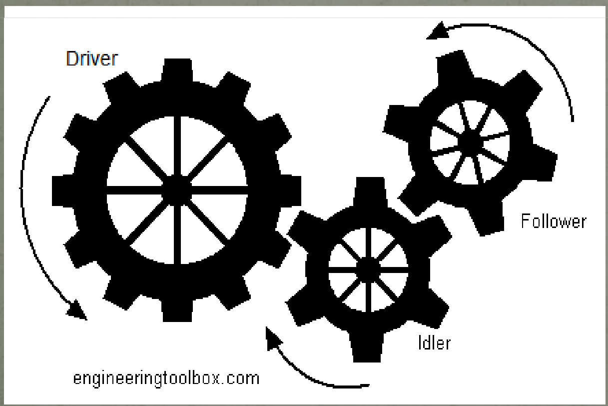


Why Gearing is necessary?

- A typical DC motor operates at speeds that are far too high to be useful, and at torques that are far too low.
- *Gear reduction* is the standard method by which a motor is made useful.

Gear Trains





Gear Ratio

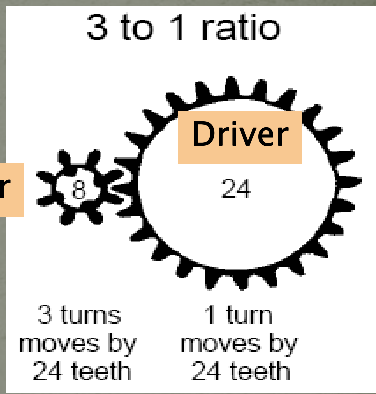
- You can calculate the **gear ratio** by using the number of teeth of the ***driver*** divided by the number of teeth of the ***follower***.

- We ***gear up*** when we increase velocity and decrease torque.

Ratio: 3:1

- We ***gear down*** when we increase torque and reduce velocity.

Ratio= 1:3



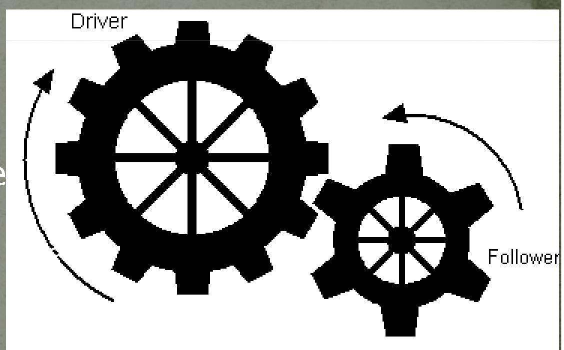
Gear Ratio = # teeth input gear / # teeth output gear
= torque in / torque out = speed out / speed in

Mathematical Modeling of Gear Trains

- Gears increase or reduce angular velocity (while simultaneously decreasing or increasing torque, such that energy is conserved).

Energy of Driving Gear = Energy of Following Gear

$$N_1\theta_1 = N_2\theta_2$$



N_1 → Number of Teeth of Driving Ge

θ_1 → Angular Movement of Driving

N_2 → Number of Teeth of Following

θ_2 → Angular Movement of Following Gear

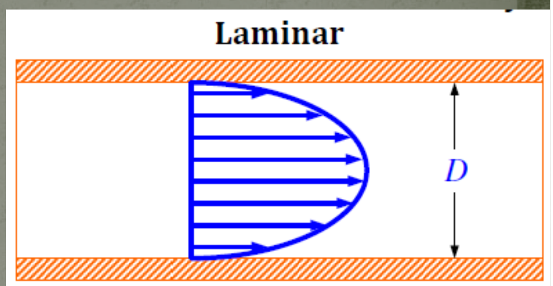
Self Study

Electro-Mechanical Systems.

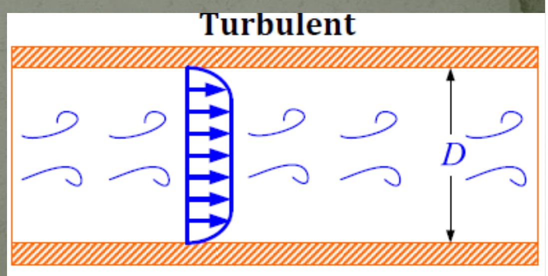
Liquid Level Systems.

Laminar *vs* Turbulent Flow

- **Laminar Flow**
  - Flow dominated by viscosity forces is called laminar flow and is characterized by a smooth, parallel line motion of the fluid

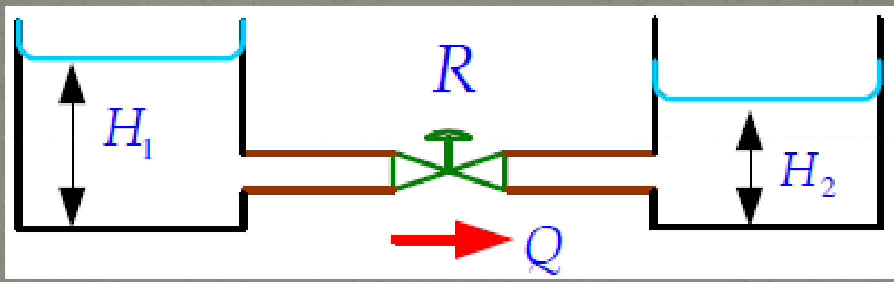


- **Turbulent Flow**
  - When inertia forces dominate, the flow is called turbulent flow and is characterized by an irregular motion of the fluid.



Resistance of Liquid-Level Systems

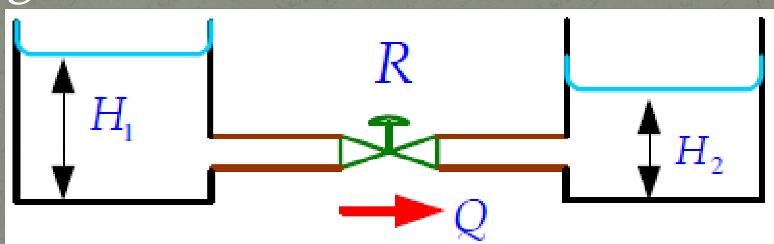
- Consider the flow through a short pipe connecting two tanks as shown in Figure.



- Where H_1 is the height (or level) of first tank, H_2 is the height of second tank, R is the resistance in flow of liquid and Q is the flow rate.

Resistance of Liquid-Level Systems

- The resistance for liquid flow in such a pipe is defined as the change in the level difference necessary to cause a unit change inflow rate.



$$Resistance = \frac{change\ in\ level\ difference}{change\ in\ flow\ rate} = \frac{m}{m^3/s}$$

$$R = \frac{\Delta(H_1 - H_2)}{\Delta Q} = \frac{m}{m^3/s}$$

Resistance in Laminar Flow

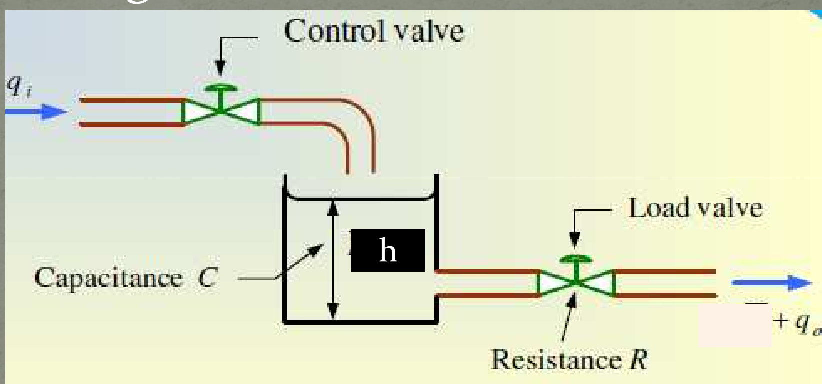
- For laminar flow, the relationship between the steady-state flow rate and steady state height at the restriction is given by:

$$Q = k_l H$$

- Where **Q** = steady-state liquid flow rate in **m³/s**
- ***K**$_l$ = constant in **m²/s***
- *and **H** = steady-state* height in ***m***.
- The resistance **R**$_l$ is $R_l = \dfrac{dH}{dQ}$

Capacitance of Liquid-Level Systems

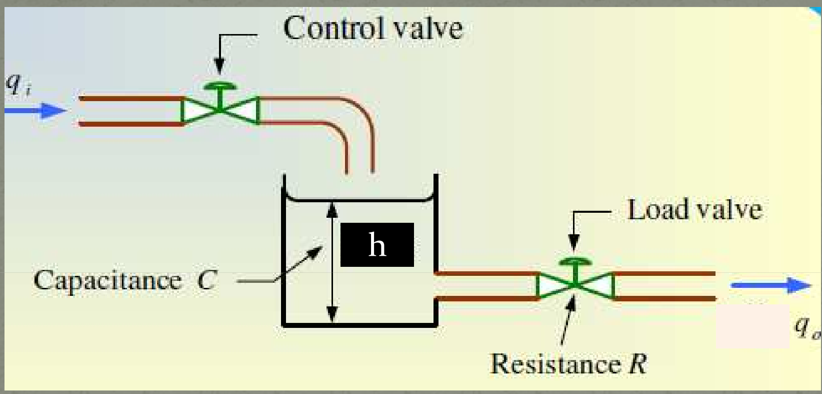
- The capacitance of a tank is defined to be the change in quantity of stored liquid necessary to cause a unity change in the height.



$$Capacitance = \frac{change\ in\ liquid\ stored}{change\ in\ height} = \frac{m^3}{m}\ or\ m^2$$

- Capacitance (***C***) is cross sectional area (***A***) of the tank.

Capacitance of Liquid-Level Systems

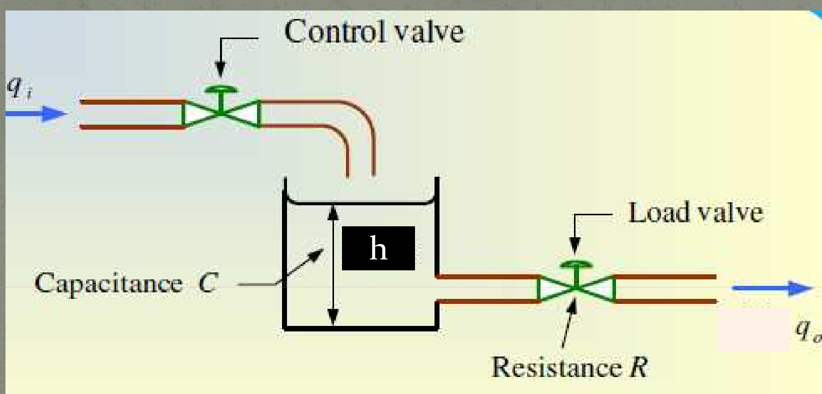


Rate of change of fluid volume in the tank=flow in-flow out

$$\frac{dV}{dt} = q_i - q_o$$

$$\frac{d(A \times h)}{dt} = q_i - q_o$$

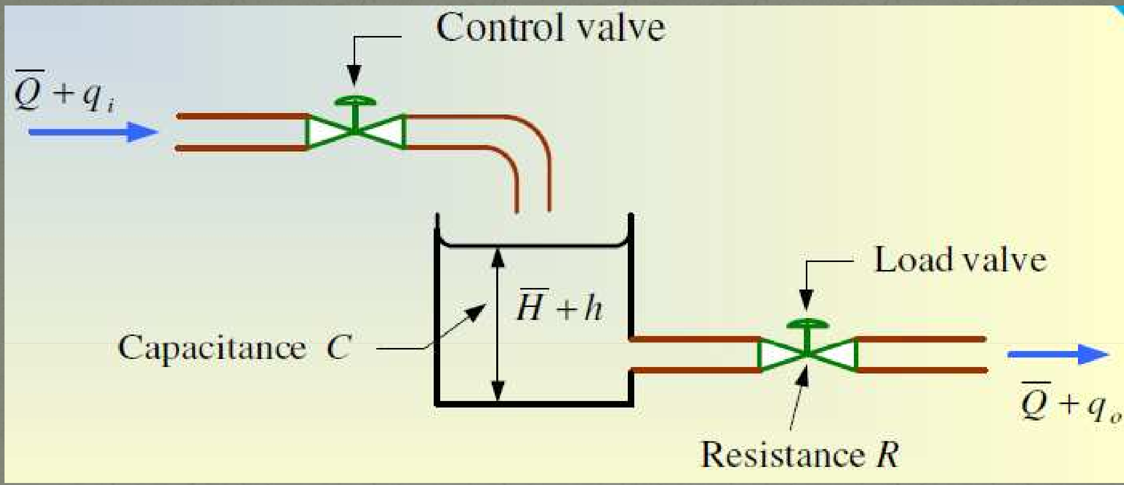
Capacitance of Liquid-Level Systems



$$A \frac{dh}{dt} = q_i - q_o$$

$$\boxed{C \frac{dh}{dt} = q_i - q_o}$$

Modelling Example-1



$\overline{H}=$ steady-state head (before any change has occurred), m.

$h=$ small deviation of head from its steady-state value, m.

$\overline{Q}=$ steady-state flow rate (before any change has occurred), m³/s.

$q_i=$ small deviation of inflow rate from its steady-state value, m³/s.

$q_o=$ small deviation of outflow rate from its steady-state value, m³/s.

Modelling Example -1

- The rate of change in liquid stored in the tank is equal to the flow in minus flow out.

$$C\frac{dh}{dt}=q_i-q_o \longrightarrow \quad (1)$$

- The resistance ***R*** may be written as

$$R=\frac{dH}{dQ}=\frac{h}{q_0} \longrightarrow \quad (2)$$

- Rearranging equation **(2)**

$$q_0=\frac{h}{R} \longrightarrow \quad (3)$$

Modelling Example -1

$$C\frac{dh}{dt} = q_i - q_o \qquad (1) \qquad\qquad q_0 = \frac{h}{R} \qquad (3)$$

- From eq. (3) substitute into eq. (**1**)

$$C\frac{dh}{dt} = q_i - \frac{h}{R}$$

- After simplifying above equation

$$RC\frac{dh}{dt} + h = Rq_i$$

- Taking Laplace transform considering initial conditions to zero

$$RCsH(s) + H(s) = RQ_i(s)$$

Modelling Example -1

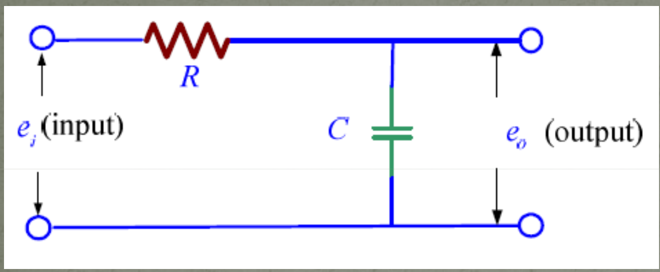
$$RCsH(s) + H(s) = RQ_i(s)$$

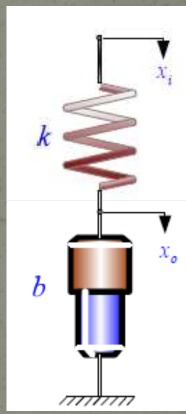
- The transfer function can be obtained as

$$\frac{H(s)}{Q_i(s)} = \frac{R}{(RCs+1)}$$

Modelling Example -1

- The liquid level system considered here is analogous to the electrical and mechanical systems shown below.





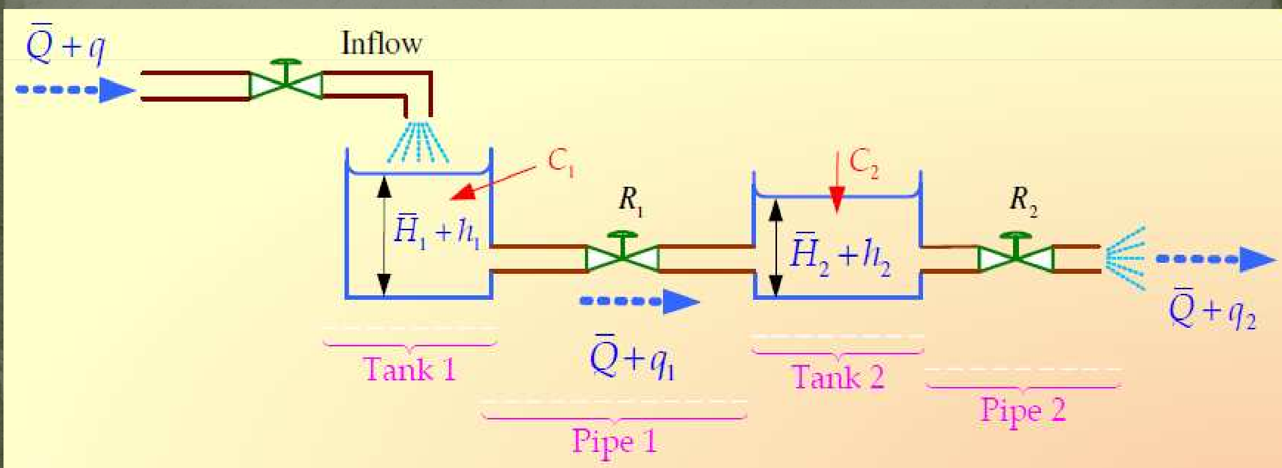
$$RC\frac{de_o}{dt}+e_o=e_i$$

$$\frac{b}{k}\frac{dx_o}{dt}+x_o=x_i$$

$$RC\frac{dh}{dt}+h=Rq_i$$

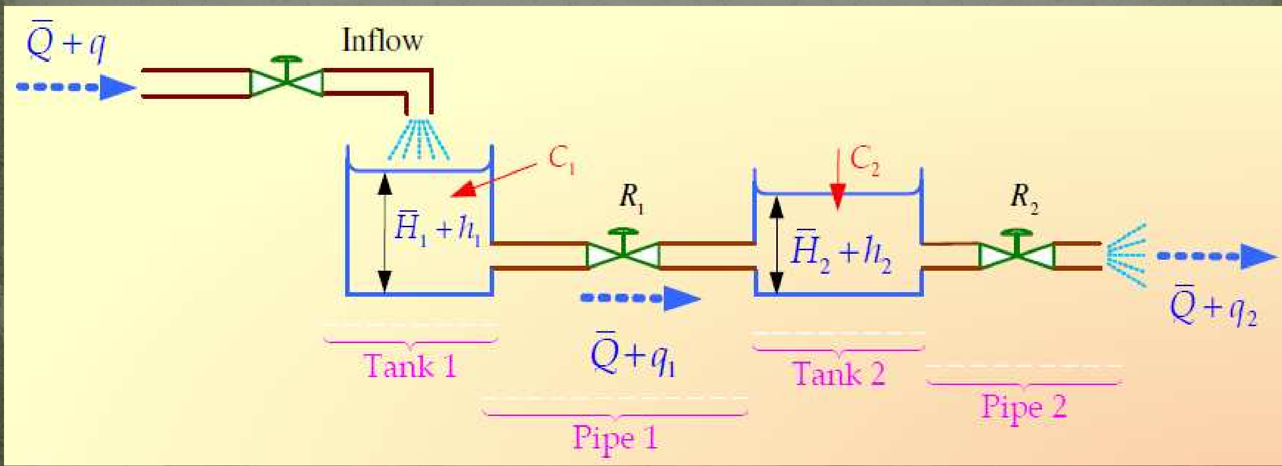
Modelling Example -2

- Consider the liquid level system shown in following Figure. In this system, two tanks interact.
- **Find** transfer function ***$Q_2(s)/Q(s)$***.



Modelling Example - 2

- Tank 1 $C_1 \frac{dh_1}{dt} = q - q_1$ $R_1 = \frac{h_1 - h_2}{q_1}$ Pipe 1
- Tank 2 $C_2 \frac{dh_2}{dt} = q_1 - q_2$ $R_2 = \frac{h_2}{q_2}$ Pipe 2



Modelling Example - 2

- Tank 1 $C_1 \frac{dh_1}{dt} = q - \frac{h_1 - h_2}{R_1}$ $q_1 = \frac{h_1 - h_2}{R_1}$ Pipe 1
- Tank 2 $C_2 \frac{dh_2}{dt} = \frac{h_1 - h_2}{R_1} - \frac{h_2}{R_2}$ $q_2 = \frac{h_2}{R_2}$ Pipe 2
- Re-arranging above equation

$$C_1 \frac{dh_1}{dt} + \frac{h_1}{R_1} = q + \frac{h_2}{R_1}$$

$$C_2 \frac{dh_2}{dt} + \frac{h_2}{R_1} + \frac{h_2}{R_2} = \frac{h_1}{R_1}$$

Modelling Example - 2

$$C_1\frac{dh_1}{dt}+\frac{h_1}{R_1}=q+\frac{h_2}{R_1} \qquad C_2\frac{dh_2}{dt}+\frac{h_2}{R_1}+\frac{h_2}{R_2}=\frac{h_1}{R_1}$$

- Taking LT of both equations considering initial conditions to zero [i.e. $h_1(0)=h_2(0)=0$].

$$\left(C_1s+\frac{1}{R_1}\right)H_1(s)=Q(s)+\frac{1}{R_1}H_2(s) \qquad (1)$$

$$\left(C_2s+\frac{1}{R_1}+\frac{1}{R_2}\right)H_2(s)=\frac{1}{R_1}H_1(s) \qquad (2)$$

Modelling Example - 2

$$\left(C_1s+\frac{1}{R_1}\right)H_1(s)=Q(s)+\frac{1}{R_1}H_2(s) \quad \textbf{(1)} \qquad \left(C_2s+\frac{1}{R_1}+\frac{1}{R_2}\right)H_2(s)=\frac{1}{R_1}H_1(s) \quad \textbf{(2)}$$

- From Equation **(1)**

$$H_1(s)=\frac{R_1Q(s)+H_2(s)}{R_1C_1s+1}$$

- Substitute the expression of **$H_1(s)$** into Equation **(2)**, we get

$$\left(C_2s+\frac{1}{R_1}+\frac{1}{R_2}\right)H_2(s)=\frac{1}{R_1}\left(\frac{R_1Q(s)+H_2(s)}{R_1C_1s+1}\right)$$

Modelling Example - 2

$$\left(C_2 s + \frac{1}{R_1} + \frac{1}{R_2}\right) H_2(s) = \frac{1}{R_1}\left(\frac{R_1 Q(s) + H_2(s)}{R_1 C_1 s + 1}\right)$$

- Using $H_2(s) = R_2 Q_2(s)$ in the above equation

$$\left[(R_2 C_2 s + 1)(R_1 C_1 s + 1) + R_2 C_1 s\right] Q_2(s) = Q(s)$$

$$\frac{Q_2(s)}{Q(s)} = \frac{1}{R_2 C_1 R_1 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_2 C_1)s + 1}$$

Modelling Example -3

- Write down the system differential equations.