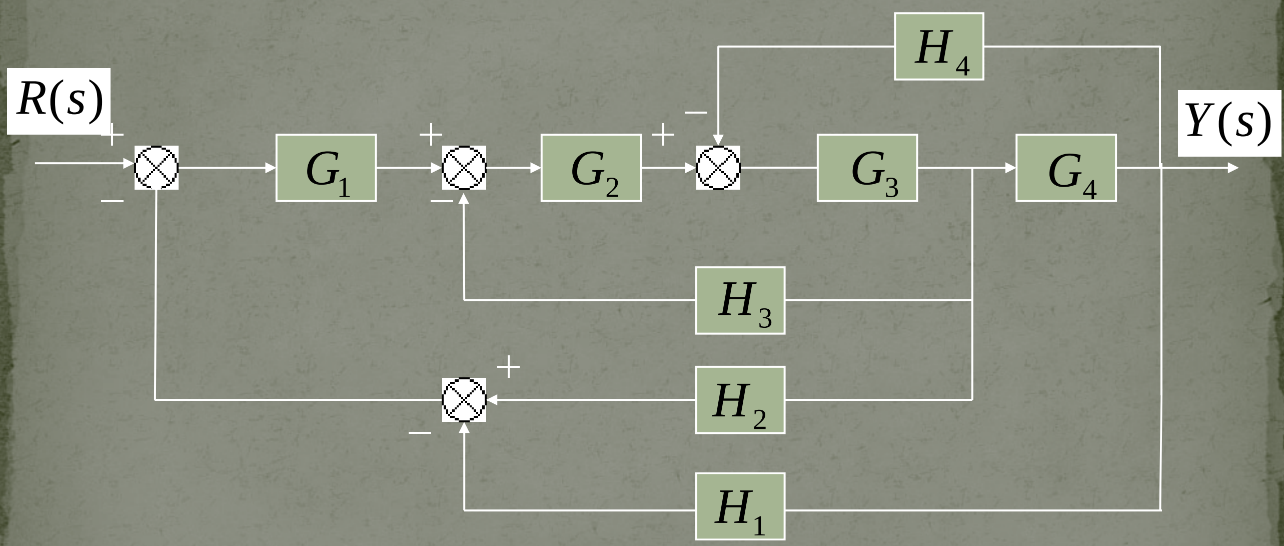
Automatic Control Theory

CSE 322

Lec. 3
Modeling of Dynamic System

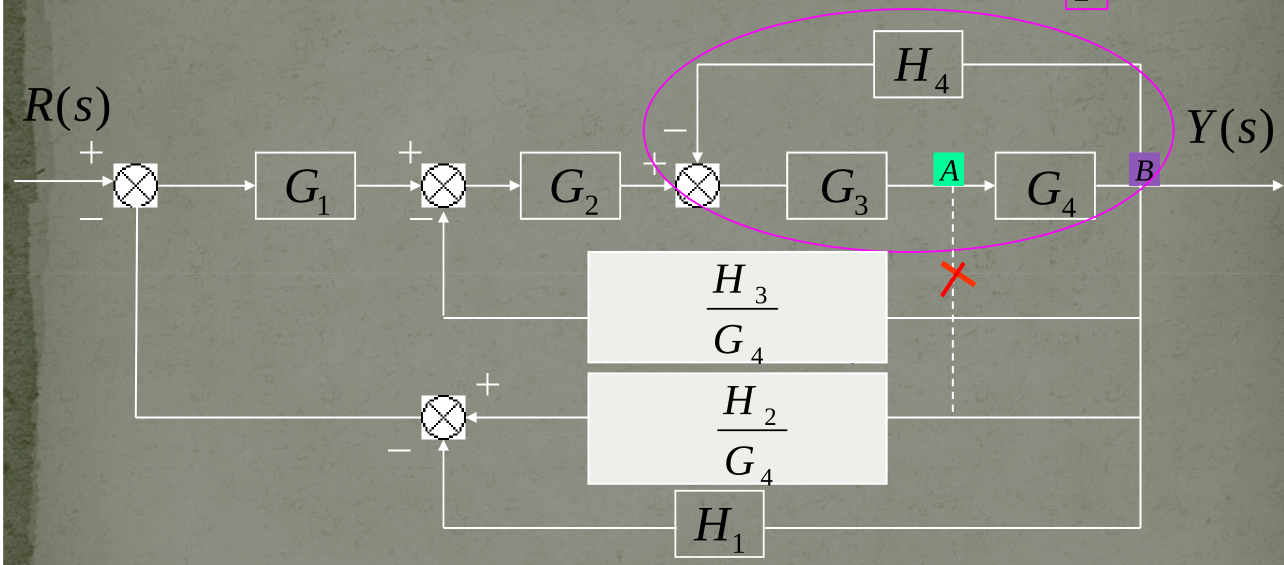
Quiz (10 mins)

Find the transfer function of the following block diagrams

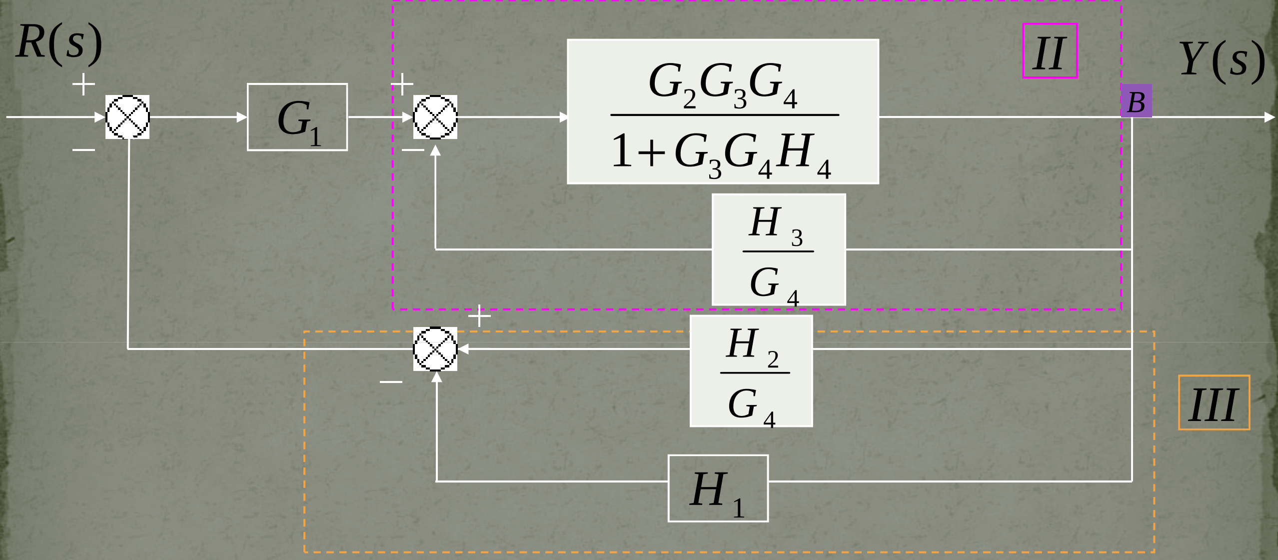


Solution:

1. Moving pickoff point A behind block G_4



2. Eliminate loop I and Simplify



II

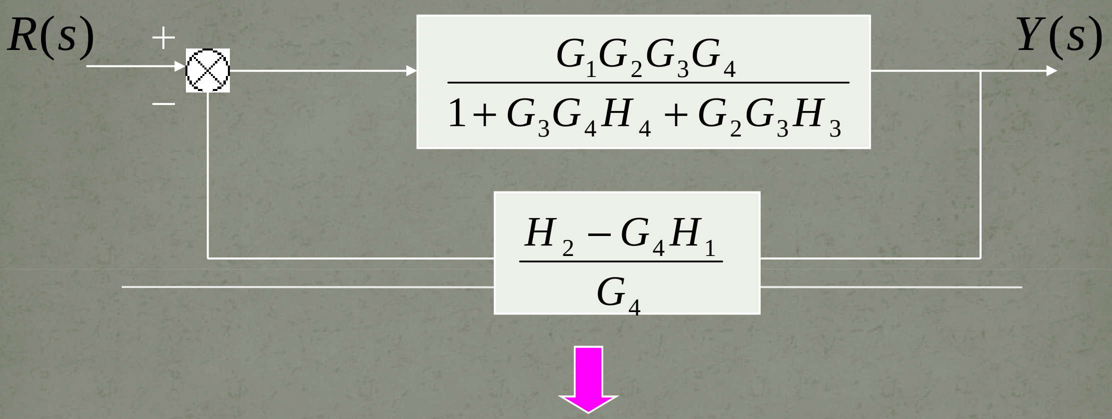
$$\frac{G_2G_3G_4}{1+G_3G_4H_4+G_2G_3H_3}$$

III

Not feedback

$$\frac{H_2-G_4H_1}{G_4}$$

3. Eliminate loop II & IIII



$$\frac{Y(s)}{R(s)}=\frac{G_1G_2G_3G_4}{1+G_2G_3H_3+G_3G_4H_4+G_1G_2G_3H_2-G_1G_2G_3G_4H_1}$$

Types of Systems

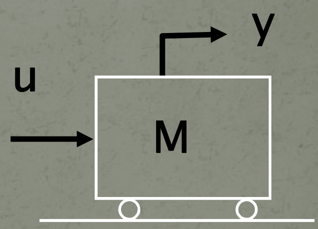
- **Static System:** If a system does not change with time, it is called a static system.
- The o/p depends on the i/p at the present time only

- **Dynamic System:** If a system changes with time, it is called a dynamic system.
- The o/p depends on the i/p at the present time and the past time. (integrator, differentiator).

Dynamic Systems

- A system is said to be dynamic if its current output may depend on the past history as well as the present values of the input variables.
- Mathematically, $y(t) = \varphi[u(\tau), 0 \le \tau \le t]$
  u : Input, t : Time

Example: A moving mass



Model: Force=Mass X Acceleration

$$M\ddot{y} = u$$

Modeling

1. **Mathematical.**
2. **State space.**

What is Mathematical Model?

A set of mathematical equations (e.g., differential equs.) that describes the input-output behavior of a system.

What is a model used for?

- Simulation
- Prediction/Forecasting
- Prognostics/Diagnostics
- Design/Performance Evaluation
- Control System Design

1. Electrical Systems

Basic Elements of Electrical Systems

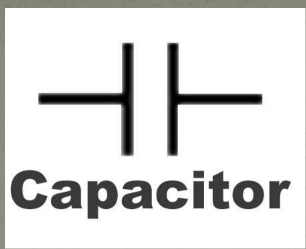
- The time domain expression relating **voltage** and **current** for the resistor is given by Ohm's law i.e.

$$v_R(t) = i_R(t)R$$

- The Laplace transform of the above equation is

$$V_R(s) = I_R(s)R$$

Basic Elements of Electrical Systems



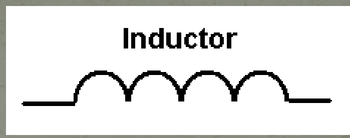
- The time domain expression relating voltage and current for the Capacitor is given as:

$$v_c(t) = \frac{1}{C}\int i_c(t)dt$$

- The Laplace transform of the above equation (assuming there is no charge stored in the capacitor) is

$$V_c(s) = \frac{1}{Cs} I_c(s)$$

Basic Elements of Electrical Systems



- The time domain expression relating voltage and current for the inductor is given as:

$$v_L(t) = L\frac{di_L(t)}{dt}$$

- The Laplace transform of the above equation (assuming there is no energy stored in inductor) is

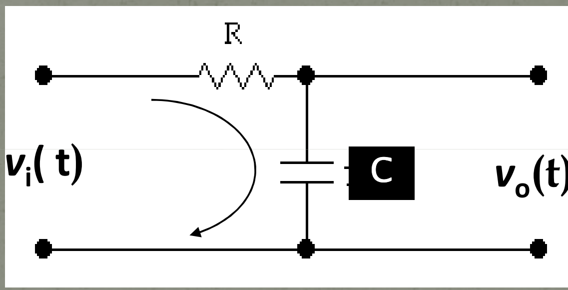
$$V_L(s) = LsI_L(s)$$

V-I and I-V relations

Component	Symbol	V-I Relation	I-V Relation
Resistor		$v_R(t) = i_R(t)R$	$i_R(t) = \frac{v_R(t)}{R}$
Capacitor		$v_c(t) = \frac{1}{C}\int i_c(t)dt$	$i_c(t) = C\frac{dv_c(t)}{dt}$
Inductor		$v_L(t) = L\frac{di_L(t)}{dt}$	$i_L(t) = \frac{1}{L}\int v_L(t)dt$

Example-1

- The two-port network shown in the following figure has $\boldsymbol{v_i}(\mathbf{t})$ as the input voltage and $\boldsymbol{v_o}(\mathbf{t})$ as the output voltage. Find the transfer function $\mathbf{V_o(s)/V_i(s)}$ of the network.



$$v_i(t) = i(t)R + \frac{1}{C}\int i(t)dt$$

$$v_o(t) = \frac{1}{C}\int i(t)dt$$

Example-1

$$v_i(t) = i(t)R + \frac{1}{C}\int i(t)dt \qquad v_o(t) = \frac{1}{C}\int i(t)dt$$

- Taking Laplace transform of both equations, considering initial conditions to zero.

$$V_i(s) = I(s)R + \frac{1}{Cs}I(s) \qquad V_o(s) = \frac{1}{Cs}I(s)$$

- Re-arrange both equations as:

$$V_i(s) = I(s)(R + \frac{1}{Cs}) \qquad CsV_o(s) = I(s)$$

Example-1

$$V_i(s) = I(s)(R + \frac{1}{Cs}) \qquad CsV_o(s) = I(s)$$

- Substitute ***I(s)*** in equation on left

$$V_i(s) = CsV_o(s)(R + \frac{1}{Cs})$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{Cs(R + \frac{1}{Cs})}$$

$$\boxed{\frac{V_o(s)}{V_i(s)} = \frac{1}{1 + RCs}}$$

Example-1

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{1 + RCs}$$

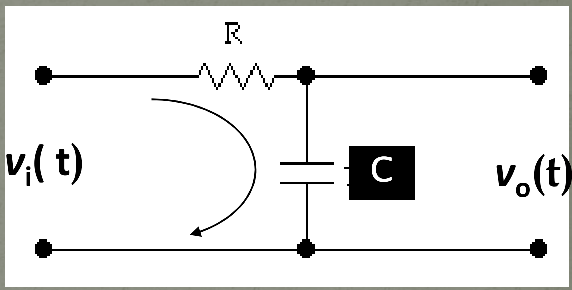
- The system has one pole at

$$1 + RCs = 0 \qquad \Rightarrow s = -\frac{1}{RC}$$

Example-2

- Design an Electrical system that would place a pole at **-4** if added to another system.

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{1 + RCs}$$



- System has one pole at

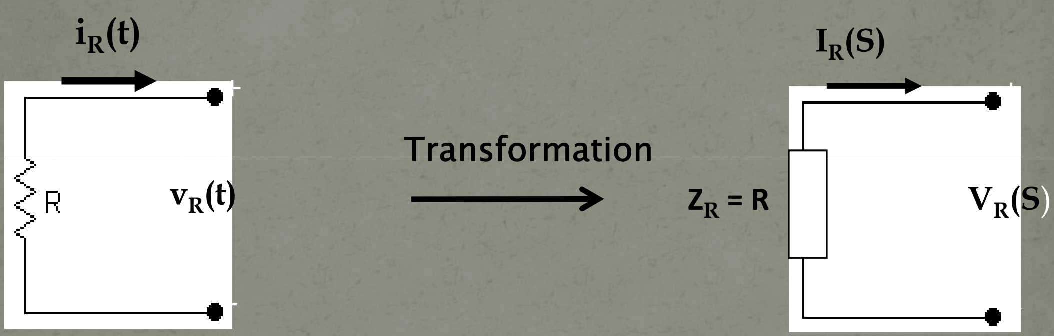
$$s = -\frac{1}{RC}$$

- Therefore,

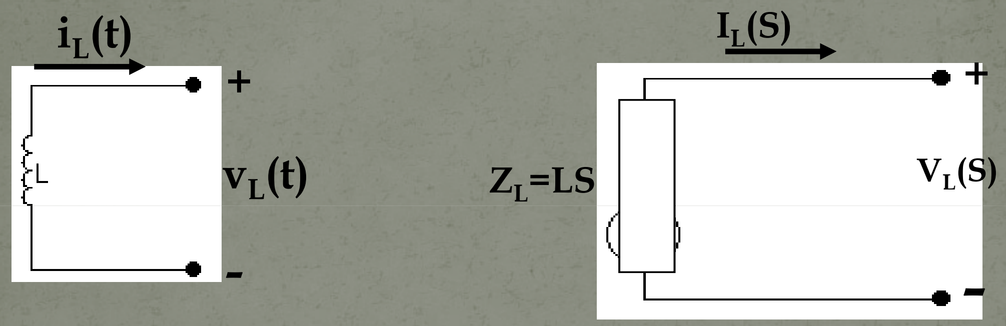
$$-\frac{1}{RC} = -4$$

if $R = 250\ \Omega$ *and* $C = 1\ \mathrm{m}F$

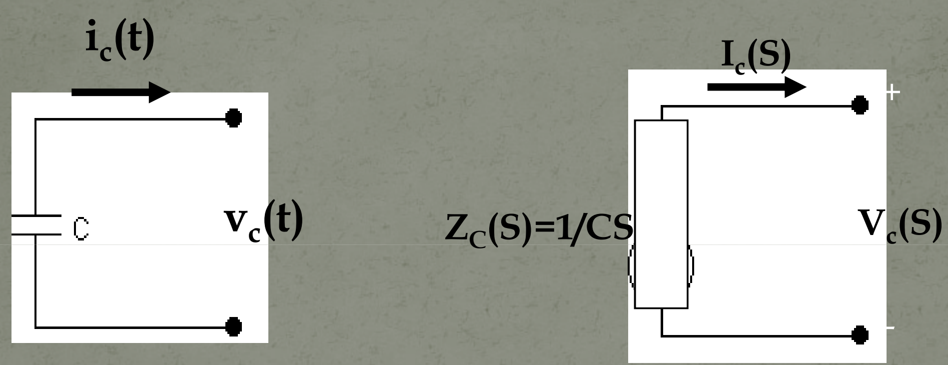
Transform Impedance (Resistor)



Transform Impedance (Inductor)



Transform Impedance (Capacitor)

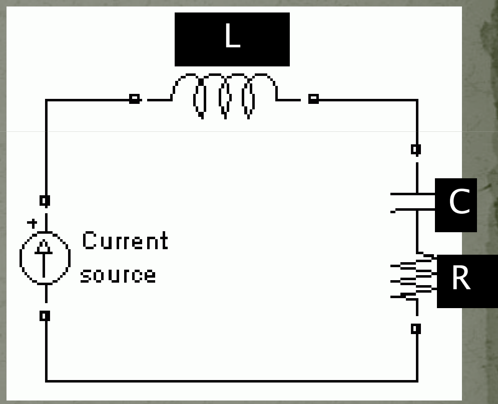


Equivalent Transform Impedance (Series)

- Consider following arrangement, find out equivalent transform impedance.

$$Z_T = Z_R + Z_L + Z_C$$

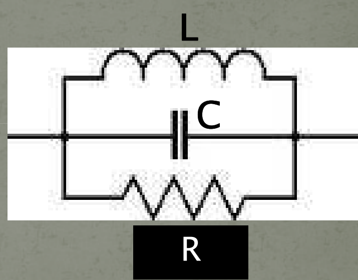
$$Z_T = R + Ls + \frac{1}{Cs}$$



Equivalent Transform Impedance (Parallel)

$$\frac{1}{Z_T} = \frac{1}{Z_R} + \frac{1}{Z_L} + \frac{1}{Z_C}$$

$$\frac{1}{Z_T} = \frac{1}{R} + \frac{1}{Ls} + \frac{1}{\frac{1}{Cs}}$$



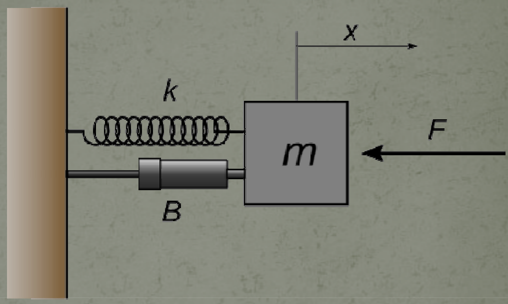
2. Mechanical Systems

Mechanical Systems

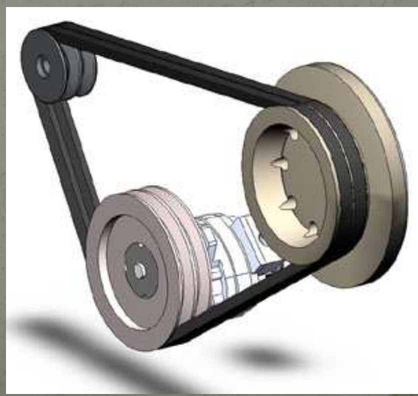
- **Part-I:** Translational Mechanical System
- **Part-II:** Rotational Mechanical System
- **Part-III:** Mechanical Linkages

Basic Types of Mechanical Systems

- **Translational**
  - Linear Motion



- **Rotational**
  - Rotational Motion

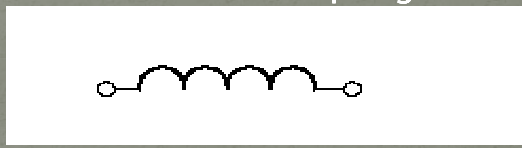


1. Translational Mechanical Systems

Basic Elements of Translational Mechanical Systems

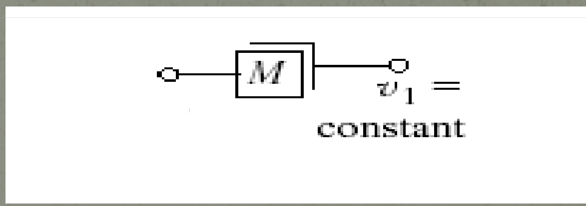
Translational Spring

i)



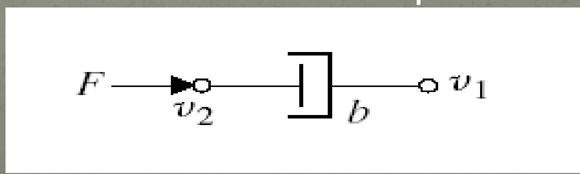
Translational Mass

ii)



Translational Damper

iii)

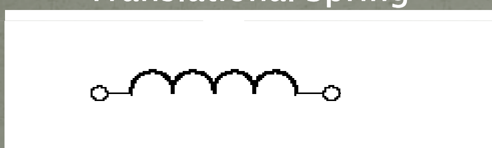


Translational Spring

- A translational spring is a mechanical element that can be deformed by an external force such that the deformation is directly <u>proportional</u> to the force applied to it.

Translational Spring

i)

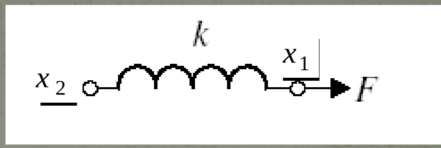


Circuit Symbols

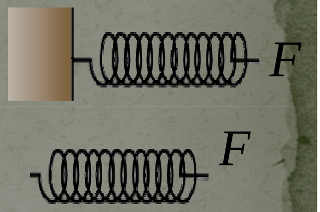
Translational Spring

Translational Spring

- If ***F*** is the applied force



- Then x_1 is the deformation if $x_2 = 0$



- Or $(x_1 - x_2)$ is the deformation.
- The equation of motion is given as

$$F = k(x_1 - x_2)$$

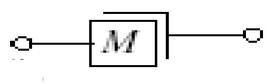
- Where k is stiffness of spring expressed in **N/m**

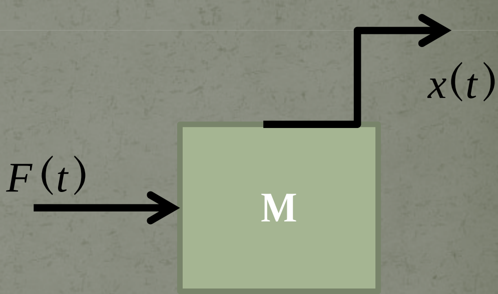
Translational Mass

- Translational Mass is an inertia element.
- A mechanical system without mass does not exist.
- If a force F is applied to a mass and it is displaced to x meters then the relation b/w force and displacements is given by Newton's law. $F = M\ddot{x}$

Translational Mass

ii)





Common Uses of Dashpots

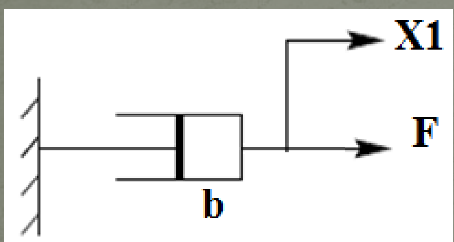
Door Stoppers

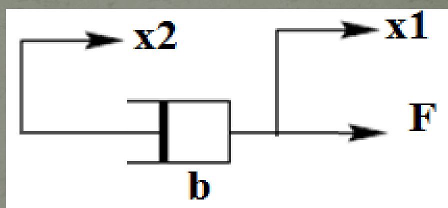
Vehicle Suspension

Bridge Suspension

Flyover Suspension

Translational Damper





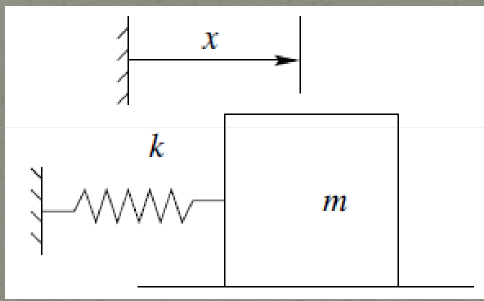
$$F = b\bar{x}$$

$$F = b(\bar{x}_1 - \bar{x}_2)$$

- Where b is damping coefficient (N/ms^{-1}).

Modeling a Simple Translational System

- **Example-1:** Consider a simple horizontal spring-mass system on a frictionless surface, as shown in figure below.



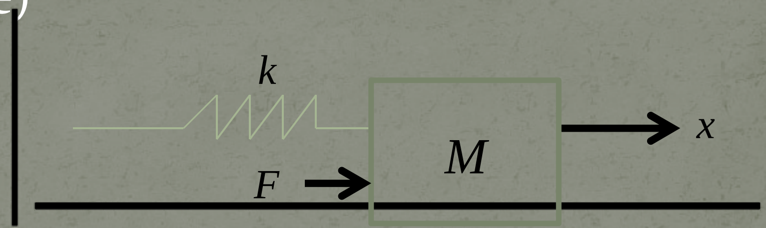
$$m\ddot{x} = -kx$$

or

$$m\ddot{x} + kx = 0$$

Example-2

- Consider the following system (friction is negligible)



- Free Body Diagram

- Where f_k and f_M are force applied by the spring and inertial force respectively.

Example-2

$$F = f_k + f_M$$

- Then the differential equation of the system is:

$$F = M\ddot{x} + kx$$

- Taking the Laplace Transform of both sides and ignoring initial conditions we get

$$F(s) = Ms^2 X(s) + kX(s)$$

Example-2

$$F(s) = Ms^2 X(s) + kX(s)$$

- The transfer function of the system is

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + k}$$

- if

$$M = 1000\ kg$$

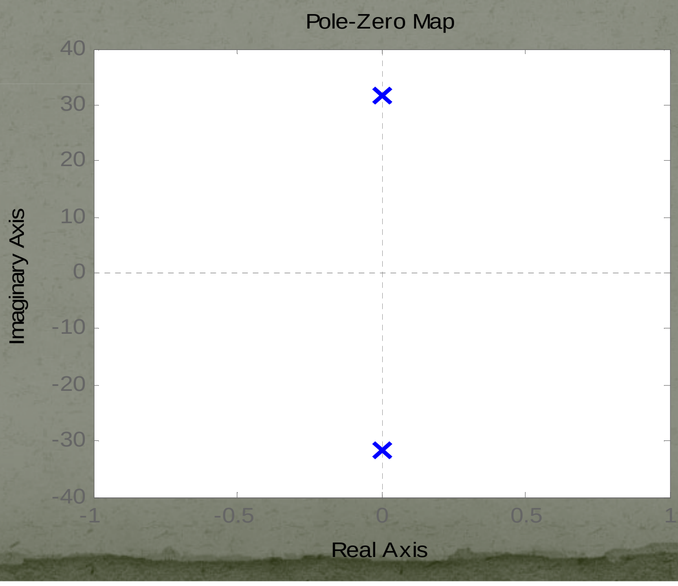
$$k = 2000\ Nm^{-1}$$

$$\frac{X(s)}{F(s)} = \frac{0.001}{s^2 + 2}$$

Example-2

$$\frac{X(s)}{F(s)} = \frac{0.001}{s^2 + 2}$$

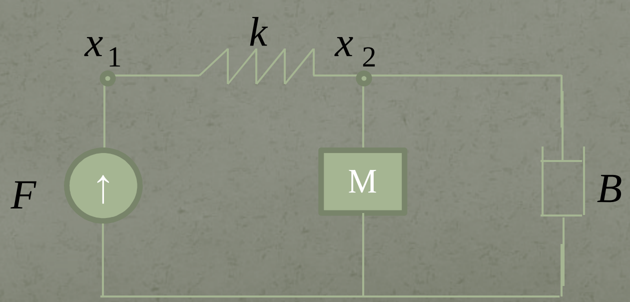
- The pole-zero map of the system is



Example-3

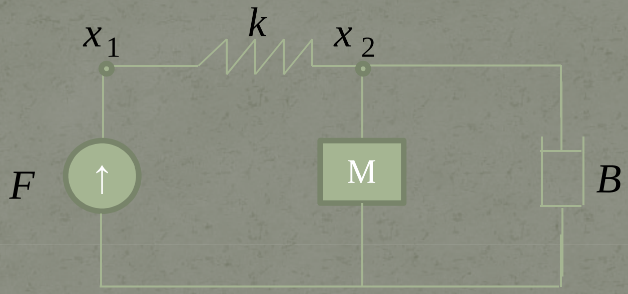
- Consider the following system

- Mechanical Network



Example-3

- Mechanical Network



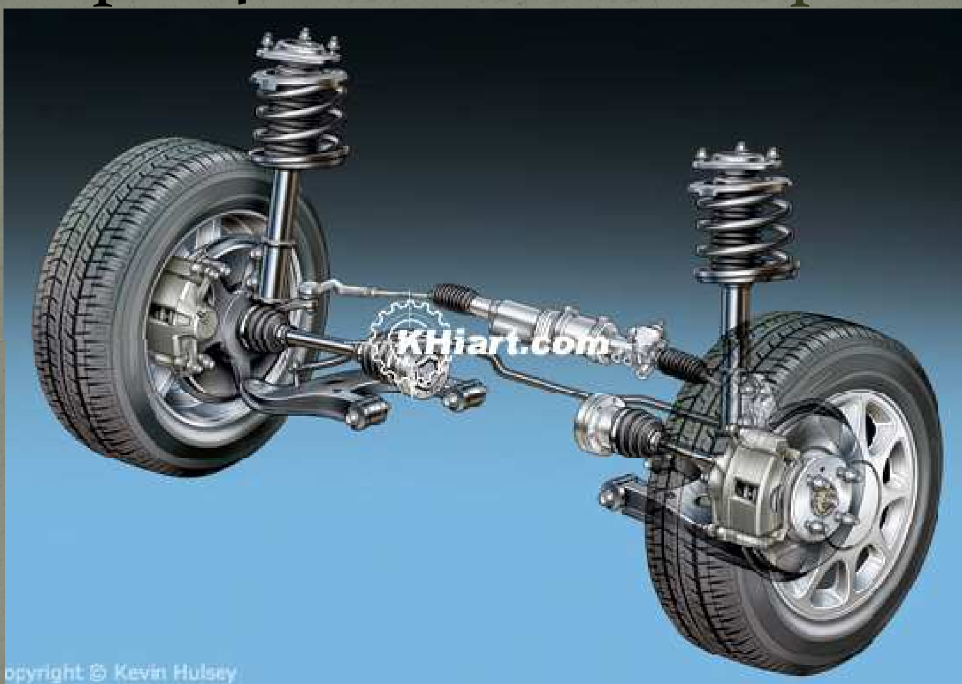
At node x_1

$$F = k(x_1 - x_2)$$

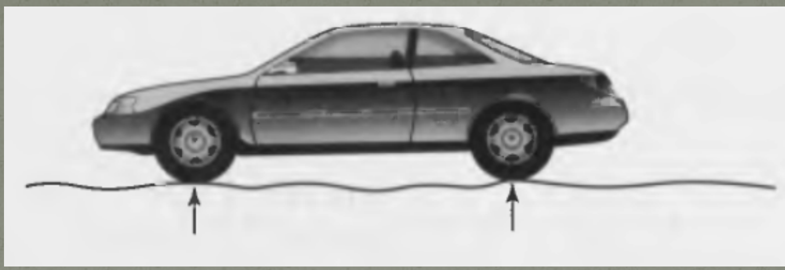
At node x_2

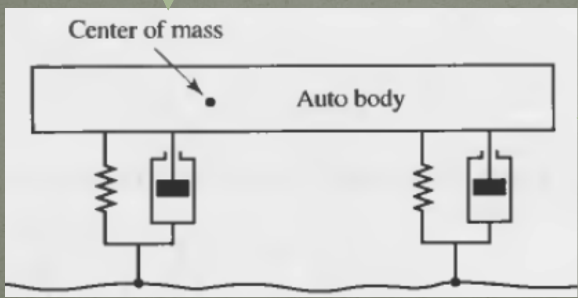
$$0 = k(x_2 - x_1) + M\ddot{x}_2 + B\dot{x}_2$$

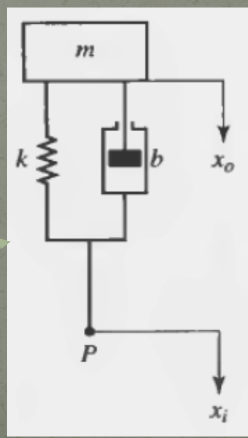
Example-4: Automobile Suspension



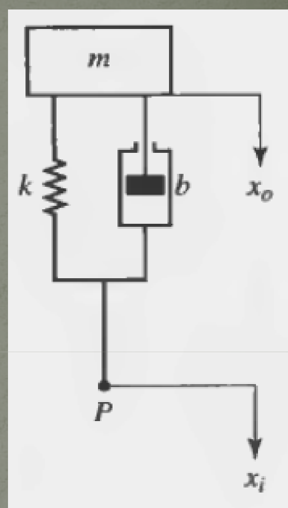
Automobile Suspension







Automobile Suspension



$$m\ddot{x}_o + b(\dot{x}_o - \dot{x}_i) + k(x_o - x_i) = 0 \qquad \text{(eq .1)}$$

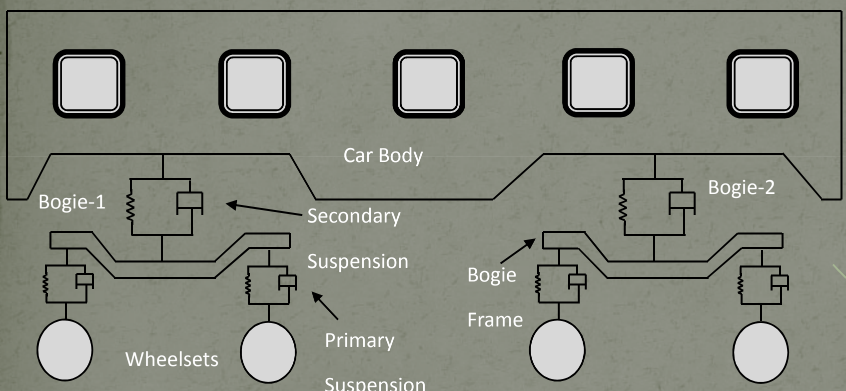
$$m\ddot{x}_o + b\dot{x}_o + kx_o = b\dot{x}_i + kx_i \qquad \text{eq. 2}$$

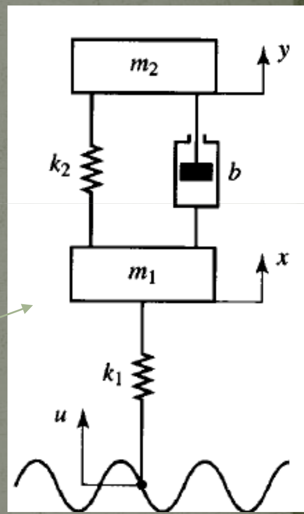
Taking Laplace Transform of the equation (2)

$$ms^2 X_o(s) + bsX_o(s) + kX_o(s) = bsX_i(s) + kX_i(s)$$

$$\frac{X_o(s)}{X_i(s)} = \frac{bs+k}{ms^2+bs+k}$$

Example-5: Train Suspension





Example-5: Train Suspension

$$m_1\ddot{x} + b\dot{x} + (k_1 + k_2)x = b\dot{y} + k_2 y + k_1 u$$
$$m_2\ddot{y} + b\dot{y} + k_2 y = b\dot{x} + k_2 x$$

Taking Laplace transforms of these two equations, assuming zero initial conditions, we obtain

$$[m_1 s^2 + bs + (k_1 + k_2)]X(s) = (bs + k_2)Y(s) + k_1 U(s)$$
$$[m_2 s^2 + bs + k_2]Y(s) = (bs + k_2)X(s)$$

Eliminating $X(s)$ from the last two equations, we have

$$(m_1 s^2 + bs + k_1 + k_2)\frac{m_2 s^2 + bs + k_2}{bs + k_2}Y(s) = (bs + k_2)Y(s) + k_1 U(s)$$

which yields

$$\frac{Y(s)}{U(s)} = \frac{k_1(bs + k_2)}{m_1 m_2 s^4 + (m_1 + m_2)bs^3 + [k_1 m_2 + (m_1 + m_2)k_2]s^2 + k_1 bs + k_1 k_2}$$