Automatic Control Theory

CSE 322

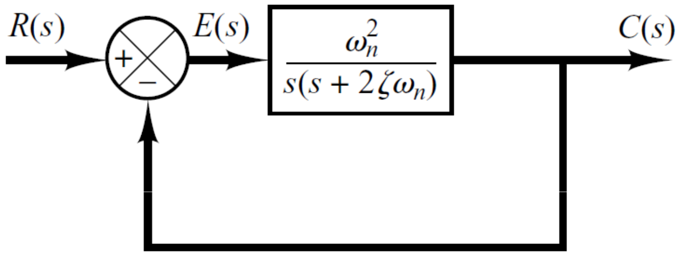
Lec. 7
Time Domain Analysis
(2^{nd} Order Systems)

Introduction

- We have already discussed the affect of location of pole and zero on the transient response of 1^{st} order systems.
- Compared to the simplicity of a first-order system, a second-order system exhibits a wide range of responses that must be analyzed and described.
- Varying a first-order system's parameters (**T, K**) simply changes the speed and offset of the response
- Whereas, changes in the parameters of a second-order system can change the ***form*** of the response.
- A second-order system can display characteristics much like a first-order system or, depending on component values, display damped or ***pure oscillations*** for its *transient response*.

Introduction

- A general second-order system (without zeros) is characterized by the following transfer function.



$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$$ **Open-Loop Transfer Function**

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$ **Closed-Loop Transfer Function**

Introduction

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

ζ ⟶ **damping ratio** of the second order system, which is a measure of the degree of resistance to change in the system output.

ω_n ⟶ **un-damped natural frequency** of the second order system, which is the frequency of oscillation of the system without damping.

Example -1

- Determine the un-damped natural frequency and damping ratio of the following second order system.

$$\frac{C(s)}{R(s)} = \frac{4}{s^2 + 2s + 4}$$

- Compare the numerator and denominator of the given transfer function with the general 2nd order transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = 4 \quad \Rightarrow \omega_n = 2\ rad/\sec$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 2s + 4$$

$$\Rightarrow 2\zeta\omega_n s = 2s$$

$$\Rightarrow \zeta\omega_n = 1$$

$$\Rightarrow \zeta = 0.5$$

Introduction

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- The closed-loop poles of the system are

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

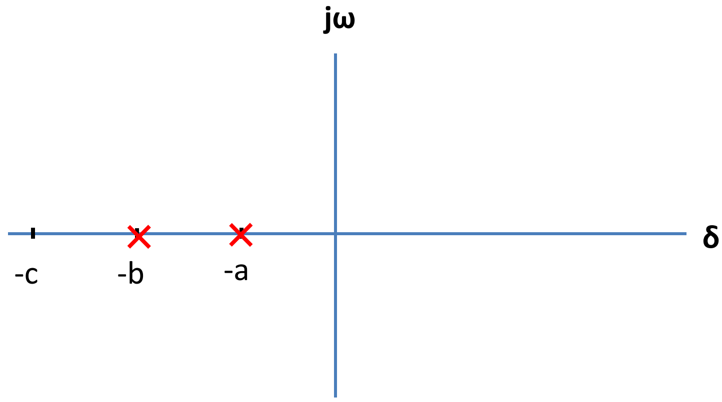
Introduction

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- Depending upon the value of ζ , a second-order system can be set into one of the four categories:

  1. Overdamped - when the system has two real distinct poles (ζ >1).



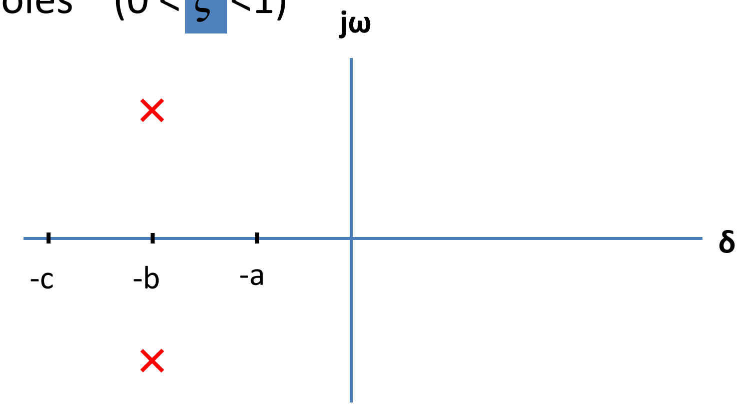
Introduction

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- According the value of ζ , a second-order system can be set into one of the four categories:

  2. *Underdamped* :- when the system has two complex conjugate poles (0 < ζ <1)



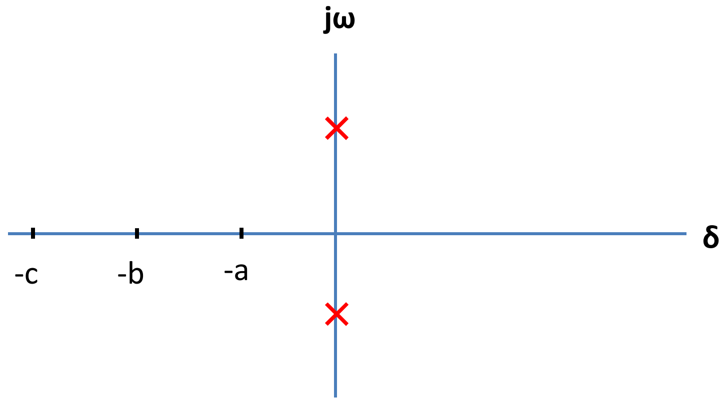
Introduction

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- According the value of ζ , a second-order system can be set into one of the four categories:

3. ***Undamped*** - when the system has two imaginary poles (ζ = 0).



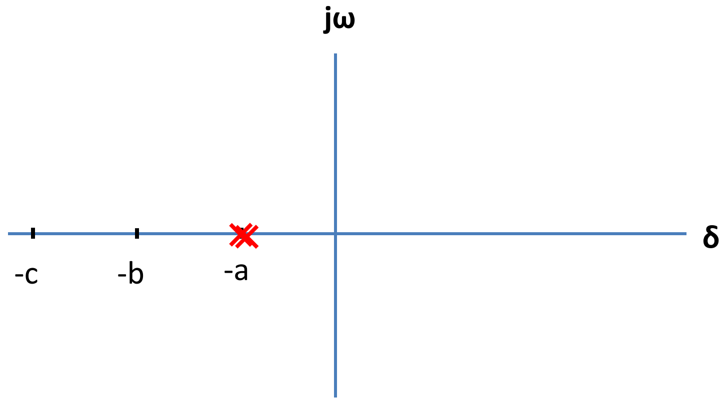
Introduction

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

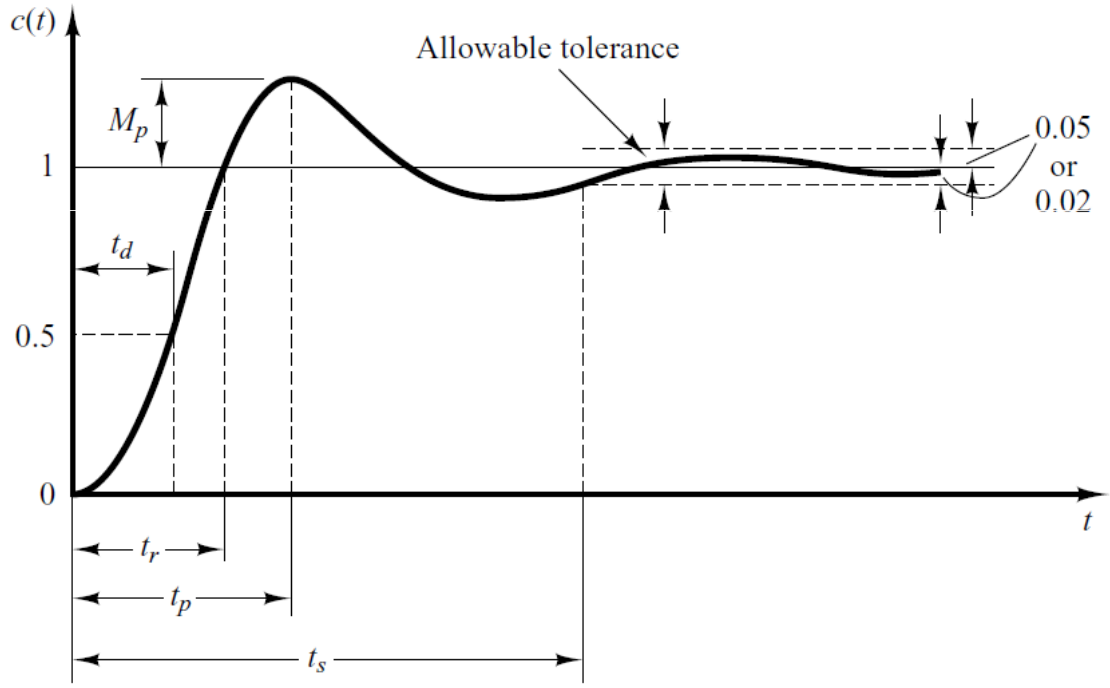
- According the value of ζ , a second-order system can be set into one of the four categories:

4. ***Critically damped*** - when the system has two real but equal poles (ζ = 1).



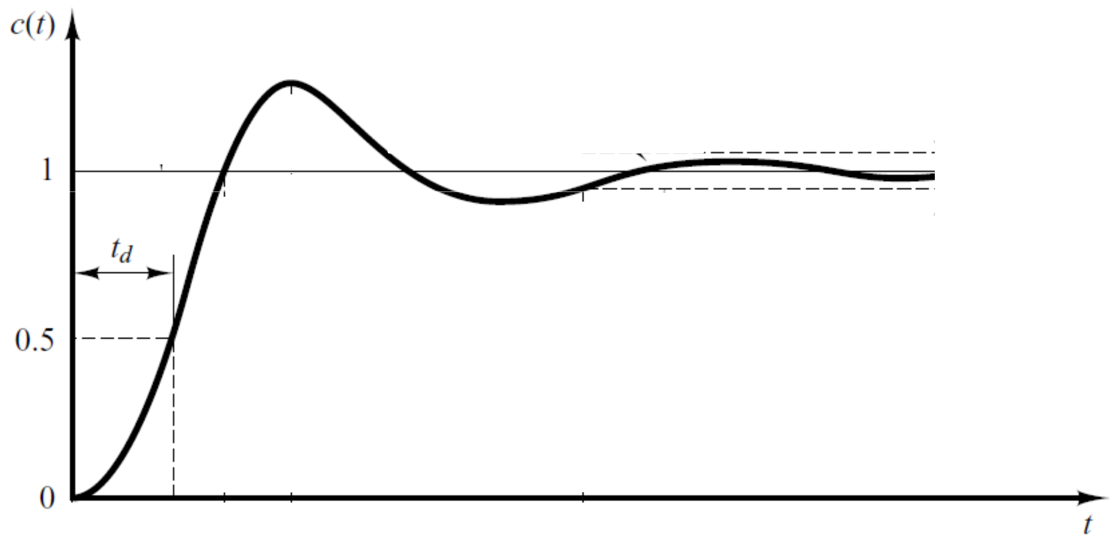
Time-Domain Specification

For $0<\zeta<1$ and $\omega_n > 0$, the 2nd order system's response due to a unit step input looks like (**underdamped**)



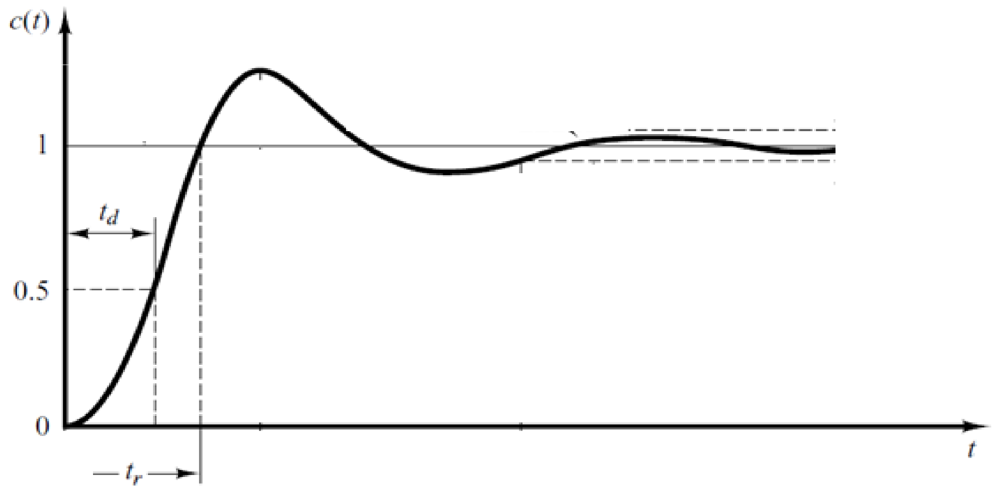
Time-Domain Specification

- The delay (t_d) time is the time required for the response to reach **half** the final value the very first time.



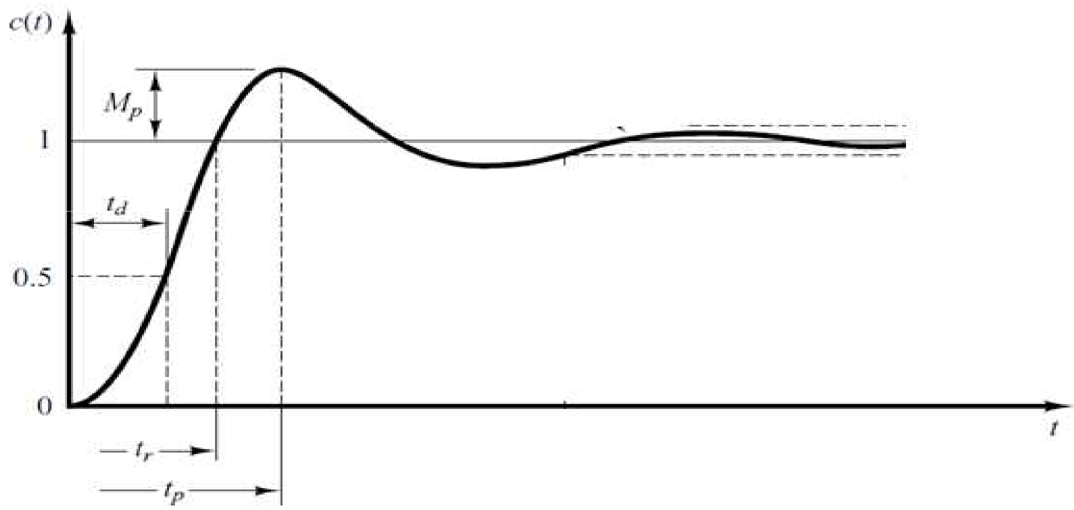
Time-Domain Specification

- **The rise time (tr)** is the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value.
- For underdamped second order systems, the 0% to 100% rise time is normally used. For overdamped systems, the 10% to 90% rise time is commonly used.



Time-Domain Specification

- **The peak time (tp)** is the time required for the response to reach the first peak of the overshoot.



Time-Domain Specification

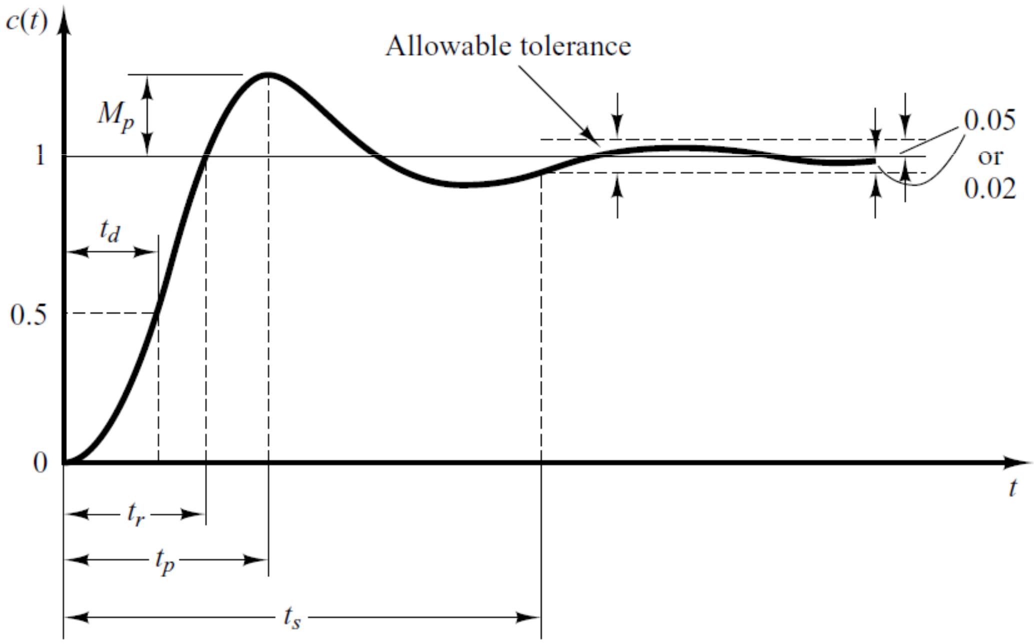
The maximum overshoot (Mp) is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

The amount of the maximum (percent) overshoot directly indicates the relative stability of the system.

Time-Domain Specification

- **The settling time (ts)** is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%).



Time Domain Specifications

Rise Time

$$t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - \theta}{\omega_n \sqrt{1 - \zeta^2}}$$

Peak Time

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}$$

Settling Time (2%)

$$t_s = 4T = \frac{4}{\zeta \omega_n}$$

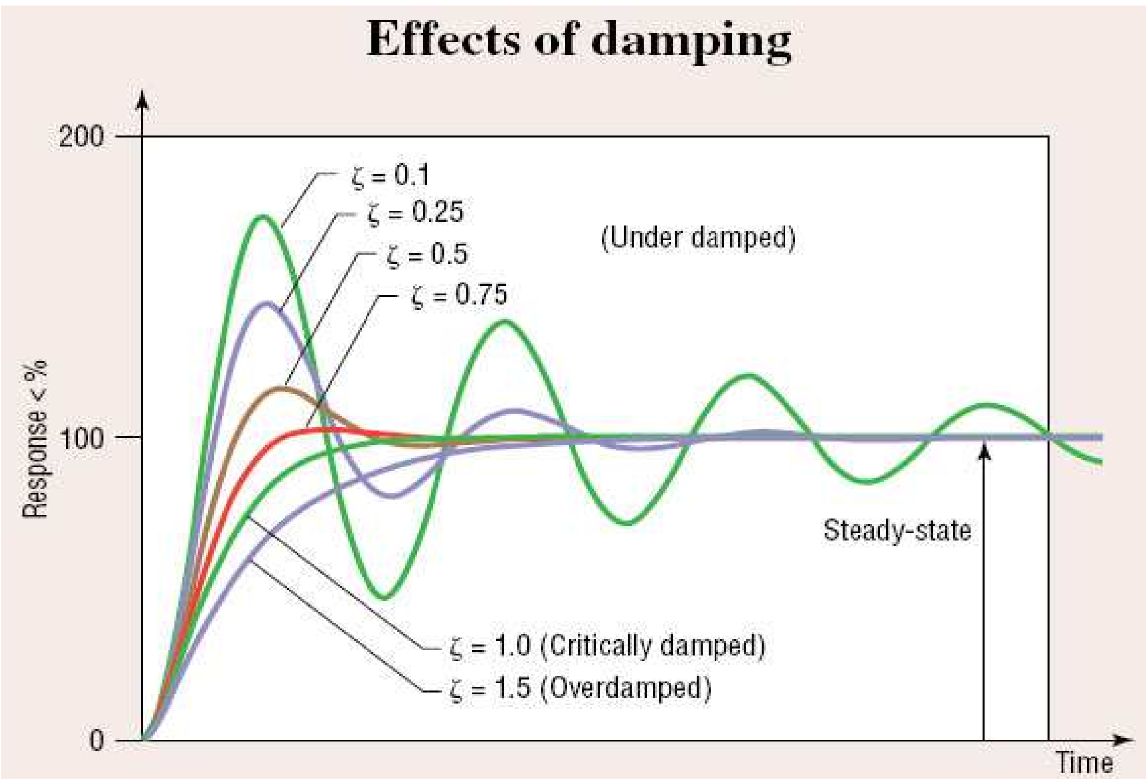
$$t_s = 3T = \frac{3}{\zeta \omega_n}$$

Settling Time (5%)

Maximum Overshoot

$$M_p = e^{-\frac{\pi \zeta}{\sqrt{1-\zeta^2}}} \times 100$$

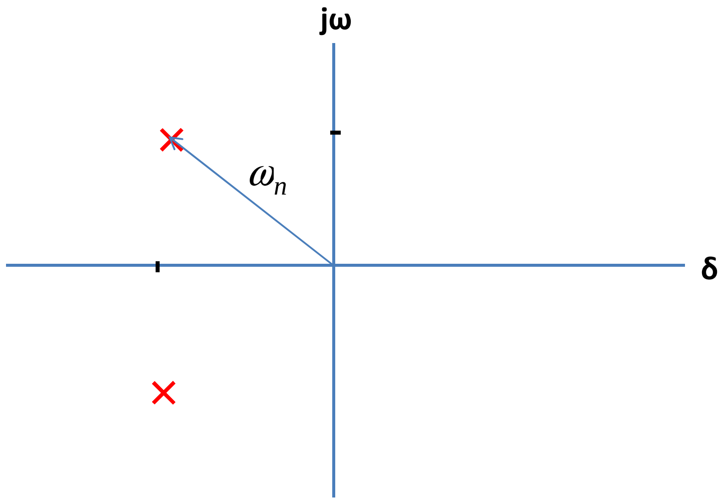
Time-Domain Specification



S-Plane

- Natural Undamped Frequency.

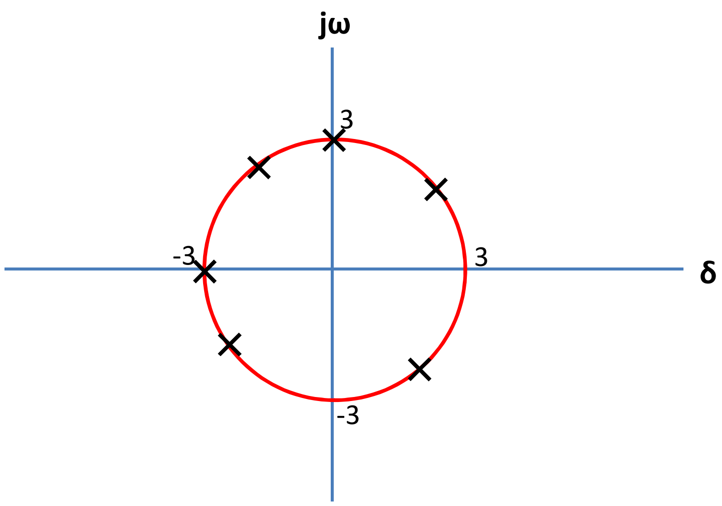
- Distance from the origin of s-plane to pole is natural undamped frequency in rad/sec.



S-Plane

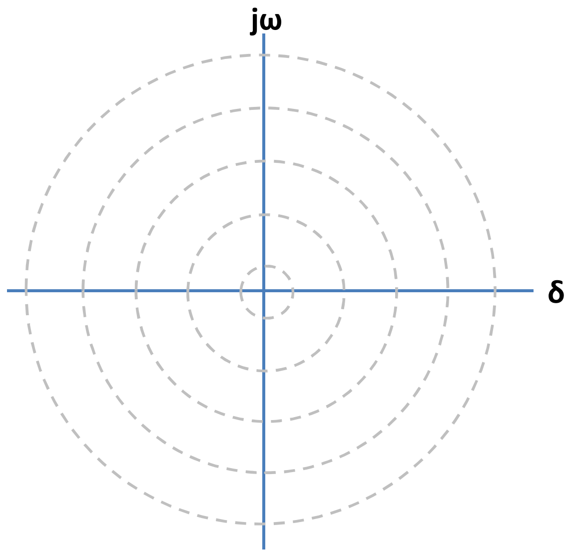
- Let us draw a circle of radius 3 in s-plane.

- If a pole is located anywhere on the circumference of the circle the natural undamped frequency would be *3 rad/sec*.



S-Plane

- Therefore the s-plane is divided into Constant Natural Undamped Frequency (ω_n) Circles.

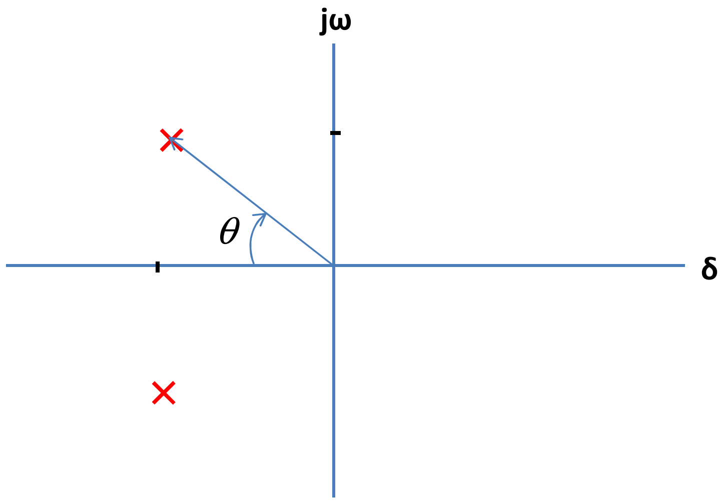


S-Plane

- **Damping ratio.**

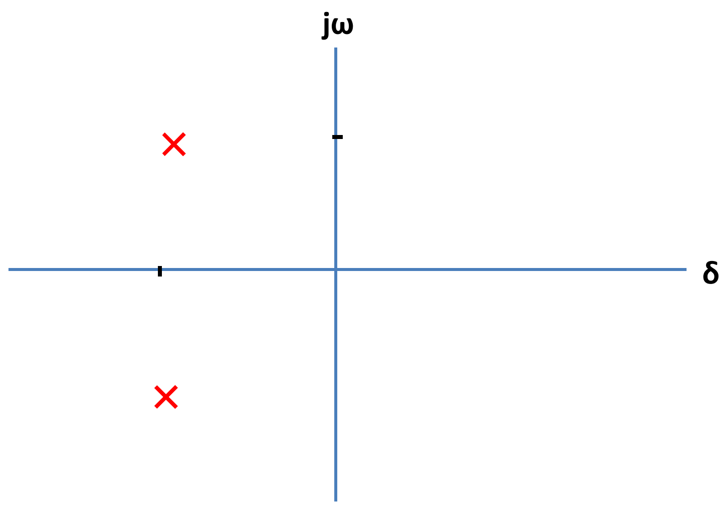
- Cosine of the angle between vector connecting origin and pole and **–ve** real axis yields damping ratio.

$$\zeta = \cos\theta$$



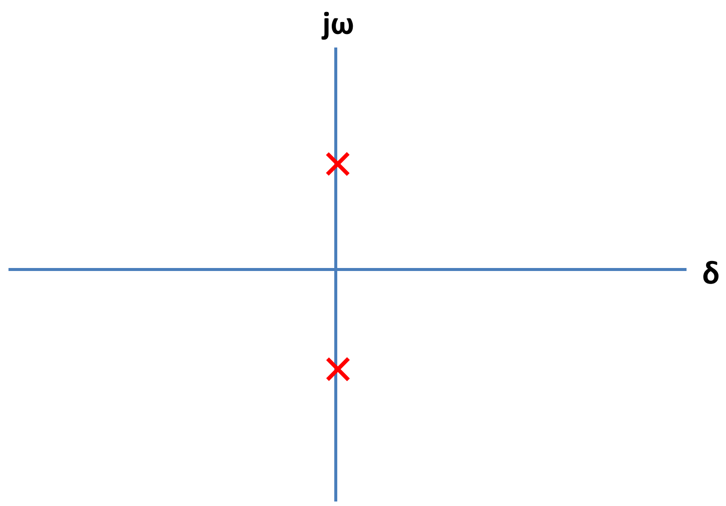
S-Plane

- For Underdamped system $0^\circ < \theta < 90^\circ$ therefore, $0 < \zeta < 1$



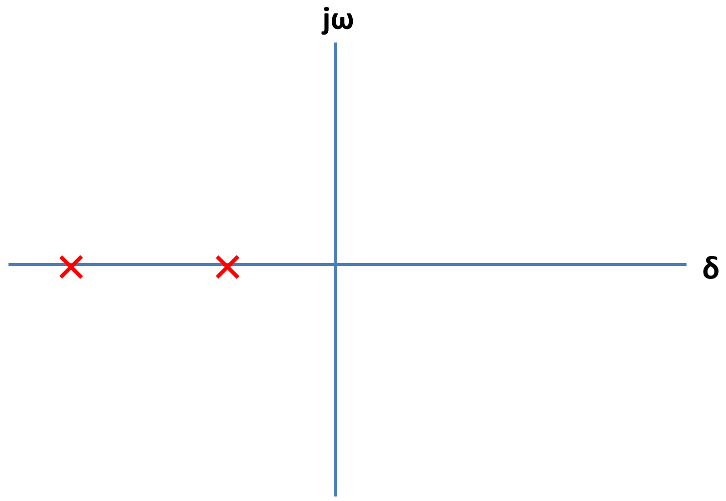
S-Plane

- For Undamped system $\theta = 90^\circ$ therefore, $\zeta = 0$



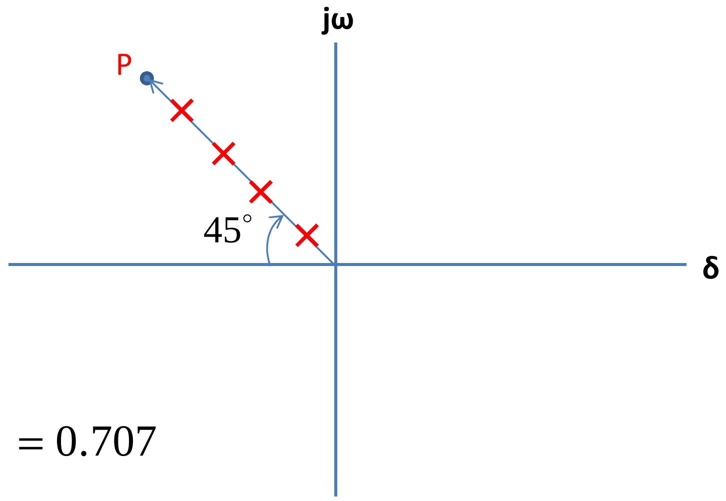
S-Plane

- For overdamped and critically damped systems $\theta = 0^{\circ}$ therefore, $\zeta \geq 1$



S-Plane

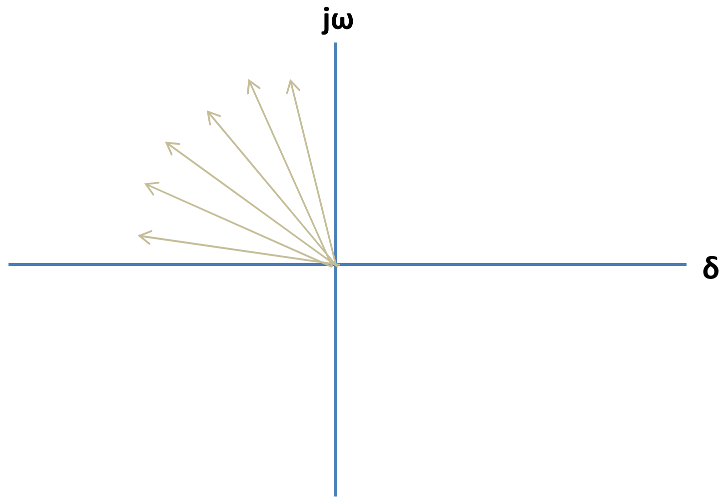
- Draw a vector connecting origin of s-plane and some point P.



$$\zeta = \cos 45^{\circ} = 0.707$$

S-Plane

- Therefore, s-plane is divided into sections of constant damping ratio lines.

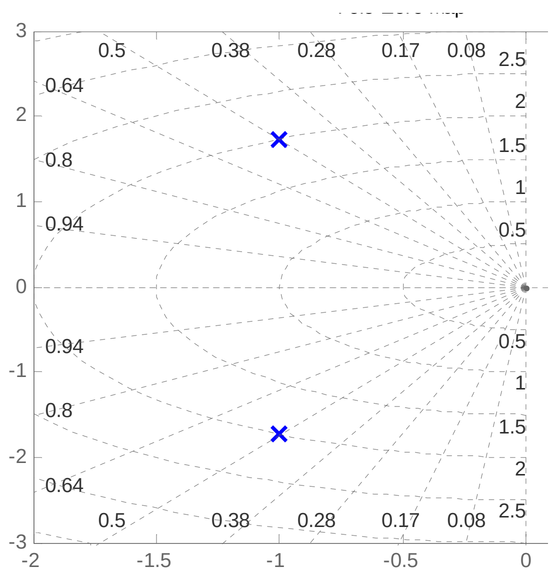


Example-2

- The natural frequency of closed loop poles of 2nd order system is 2 rad/sec and damping ratio is 0.5.

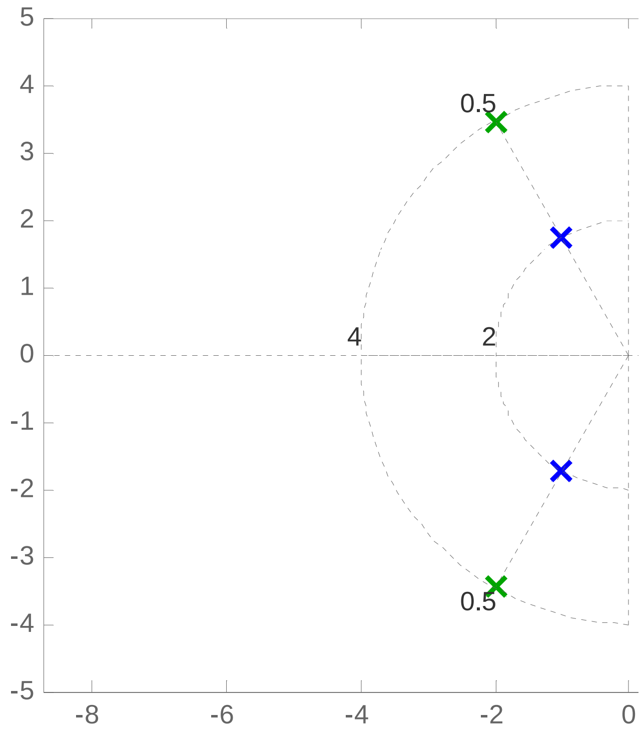
- Determine the location of closed loop poles so that the damping ratio remains same but the natural undamped frequency is doubled.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{4}{s^2 + 2s + 4}$$



Example-2

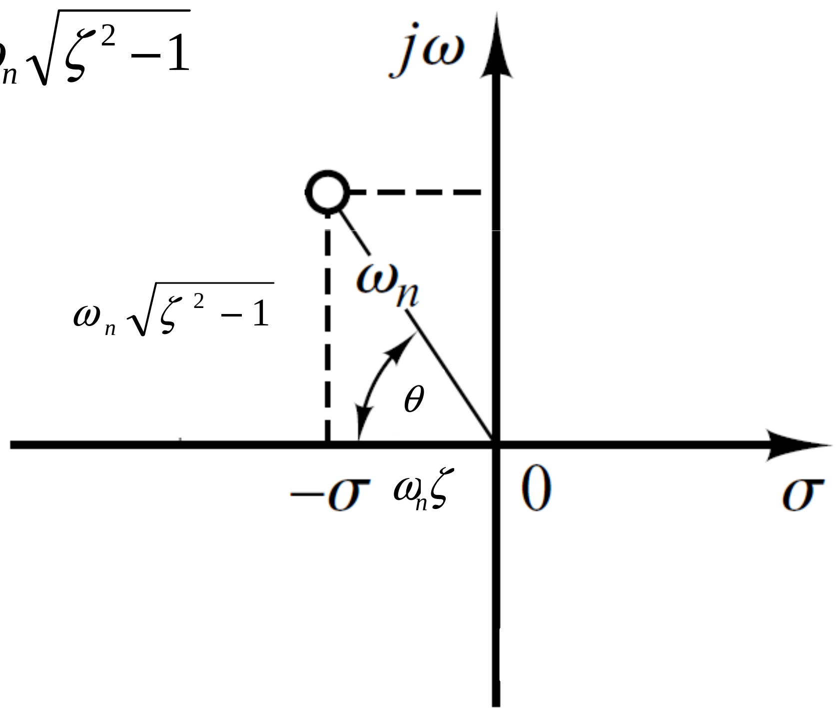
- Determine the location of closed loop poles so that the damping ratio remains **same** but the natural undamped frequency is **doubled**.



S-Plane

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$



Step Response of underdamped System

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \xrightarrow{\text{Step Response}} C(s) = \frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$

- The partial fraction expansion of above equation is given as

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2\left(1 - \zeta^2\right)$$

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2 + \omega_n^2 - \zeta^2\omega_n^2}$$

$$(s + 2\zeta\omega_n)^2$$

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2\left(1 - \zeta^2\right)}$$

Step Response of underdamped System

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2\left(1 - \zeta^2\right)}$$

- Above equation can be written as

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

- Where $\omega_d = \omega_n\sqrt{1 - \zeta^2}$, is the frequency of transient oscillations and is called **damped natural frequency.**

- The inverse Laplace transform of above equation can be obtained easily if C(s) is written in the following form:

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

Step Response of underdamped System

$$C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$$

$$C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2+\omega_d^2}$$

$$C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta}{\sqrt{1-\zeta^2}}\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_d t$$

Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_d t$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

- When $\zeta = 0$

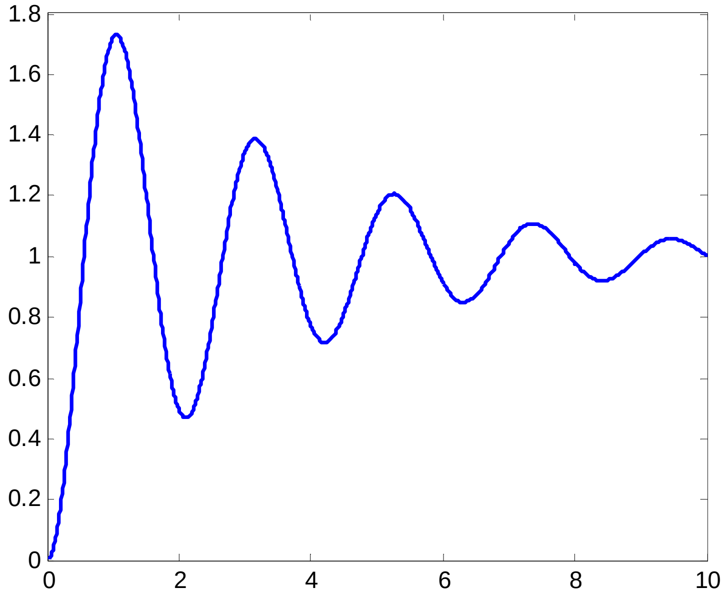
$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$
$$= \omega_n$$

$$c(t) = 1 - \cos\omega_n t$$

Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

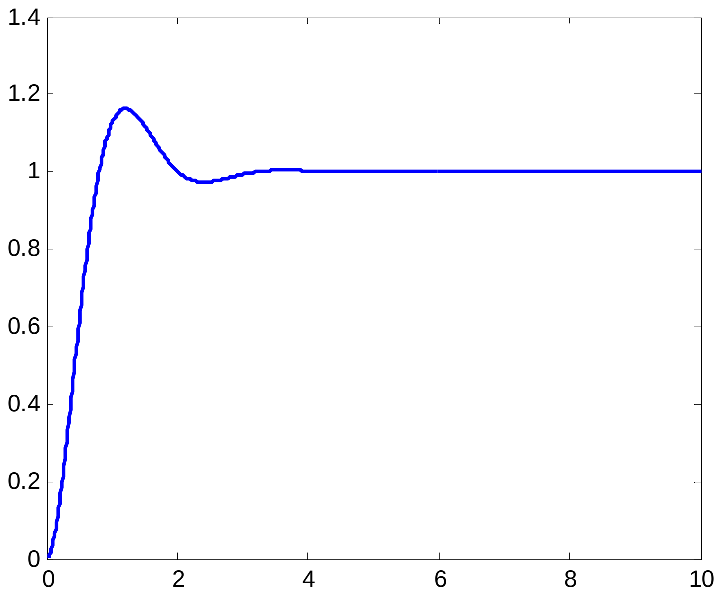
if $\zeta = 0.1$ and $\omega_n = 3 \; rad/\sec$



Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

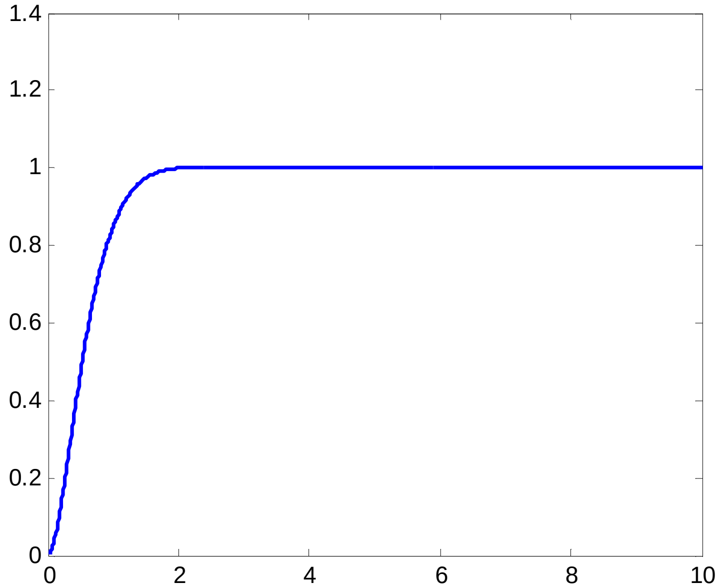
if $\zeta = 0.5$ and $\omega_n = 3 \; rad/\sec$



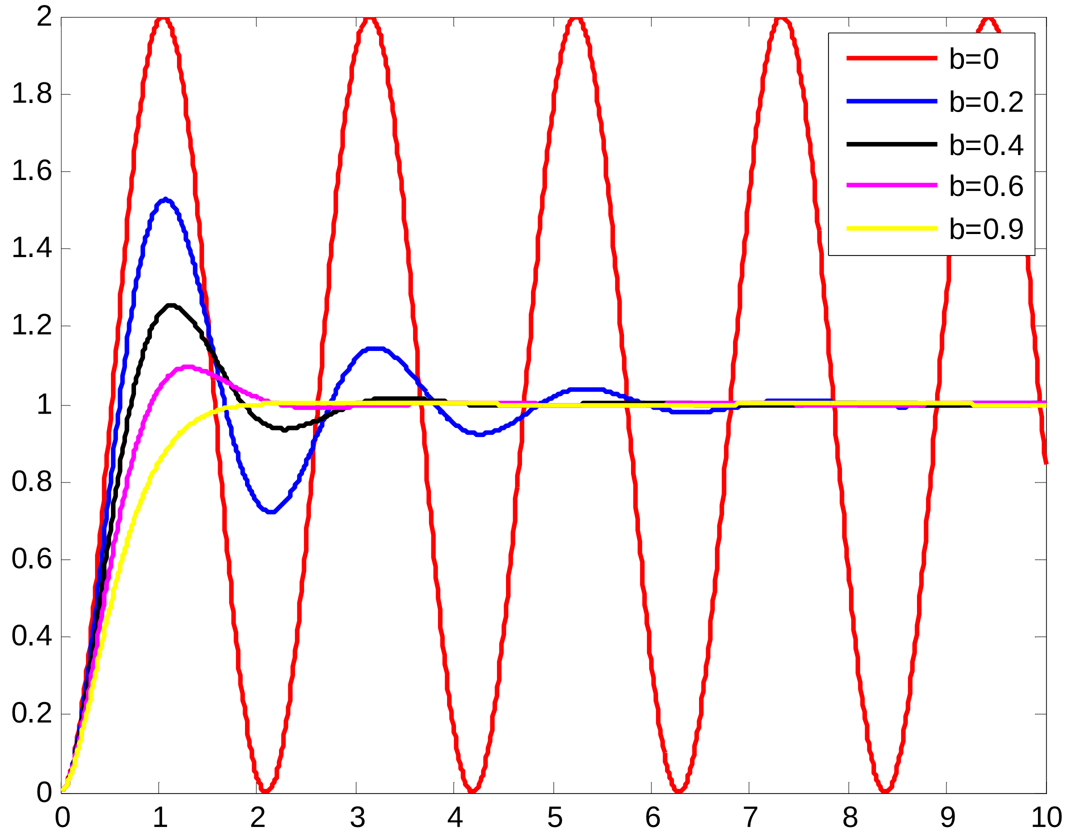
Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

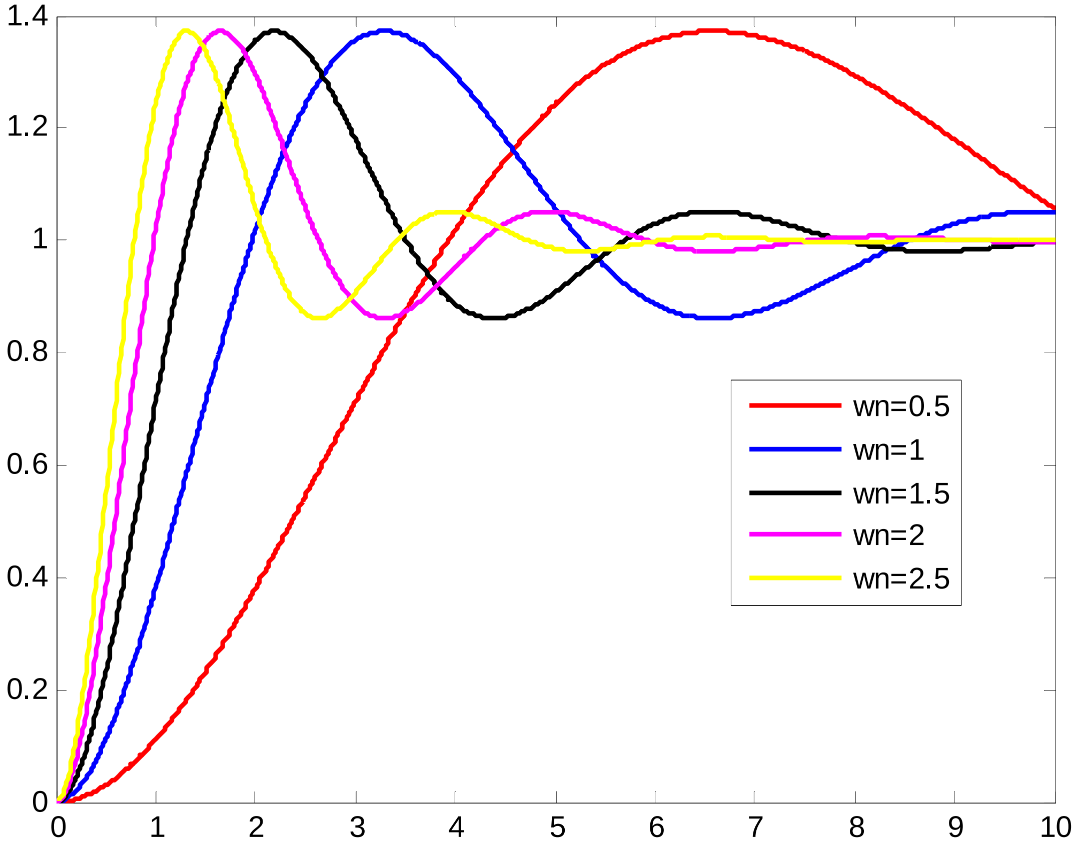
if $\zeta = 0.9$ and $\omega_n = 3 \ rad/\sec$



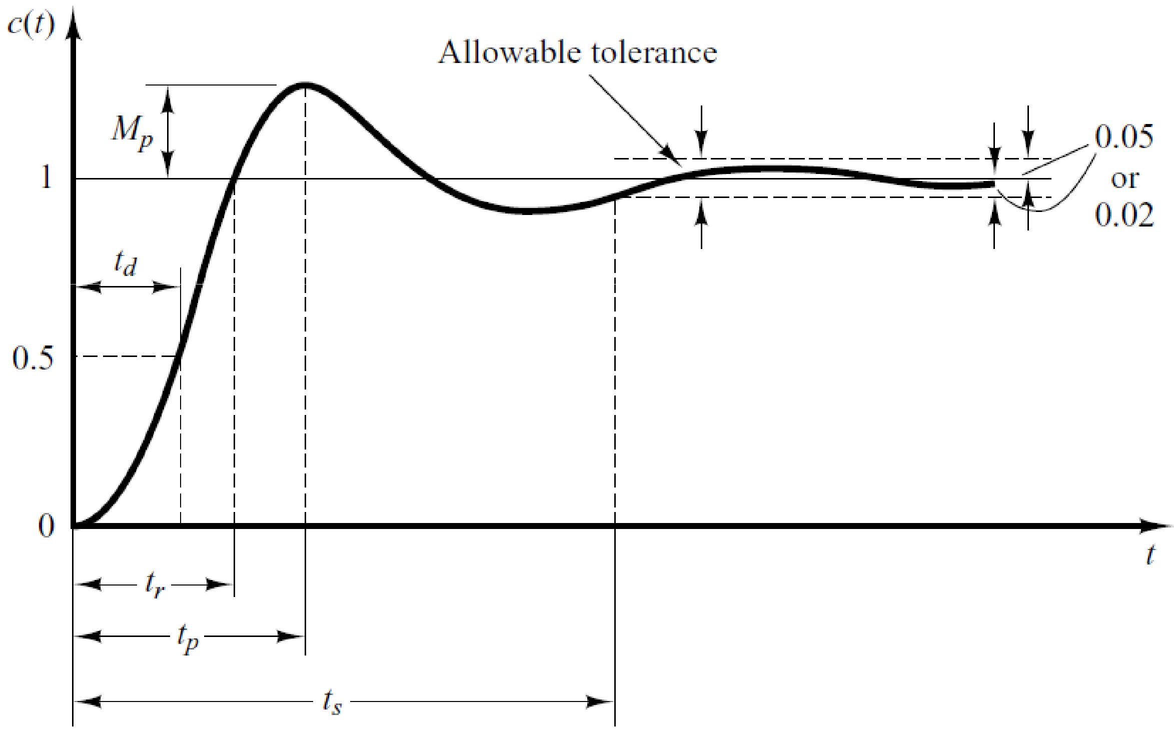
Step Response of underdamped System



Step Response of underdamped System



Time Domain Specifications of Underdamped system



Summary of Time Domain Specifications

Rise Time

$$t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - \theta}{\omega_n \sqrt{1-\zeta^2}}$$

Peak Time

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

Settling Time (2%)

$$t_s = 4T = \frac{4}{\zeta\omega_n}$$

$$t_s = 3T = \frac{3}{\zeta\omega_n}$$

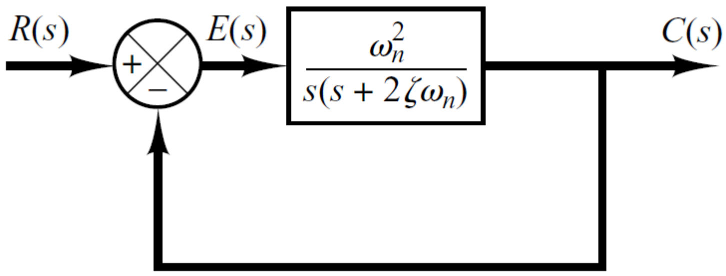
Settling Time (5%)

Maximum Overshoot

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

Example –3

- Consider the system shown in following figure, where damping ratio is 0.6 and natural undamped frequency is 5 rad/sec. Obtain the rise time t_r, peak time t_p, maximum overshoot M_p, and settling time 2% and 5% criterion t_s when the system is subjected to a unit-step input.



Example–3

Rise Time

$$t_r = \frac{\pi - \theta}{\omega_d}$$

Peak Time

$$t_p = \frac{\pi}{\omega_d}$$

Settling Time (2%)

$$t_s = 4T = \frac{4}{\zeta\omega_n}$$

$$t_s = 3T = \frac{3}{\zeta\omega_n}$$

Settling Time (5%)

Maximum Overshoot

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

Example –3

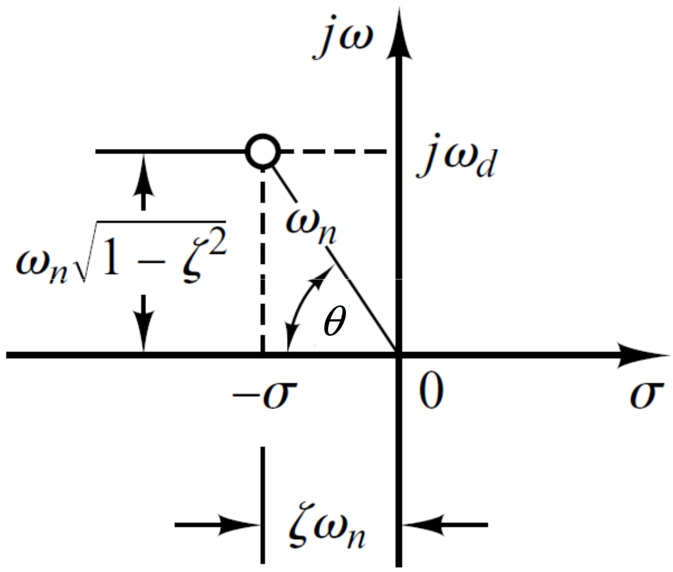
Rise Time

$$t_r = \frac{\pi - \theta}{\omega_d}$$

$$t_r = \frac{3.141 - \theta}{\omega_n\sqrt{1-\zeta^2}}$$

$$\theta = \tan^{-1}\left(\frac{\omega_n\sqrt{1-\zeta^2}}{\zeta\omega_n}\right) = 0.93\text{rad}$$

$$t_r = \frac{3.141 - 0.93}{5\sqrt{1-0.6^2}} = 0.55s$$



Example -3

Peak Time

$$t_p = \frac{\pi}{\omega_d}$$

$$t_p = \frac{3.141}{4} = 0.785s$$

Settling Time (2%)

$$t_s = \frac{4}{\zeta\omega_n}$$

$$t_s = \frac{4}{0.6 \times 5} = 1.33s$$

Settling Time (5%)

$$t_s = \frac{3}{\zeta\omega_n}$$

$$t_s = \frac{3}{0.6 \times 5} = 1s$$

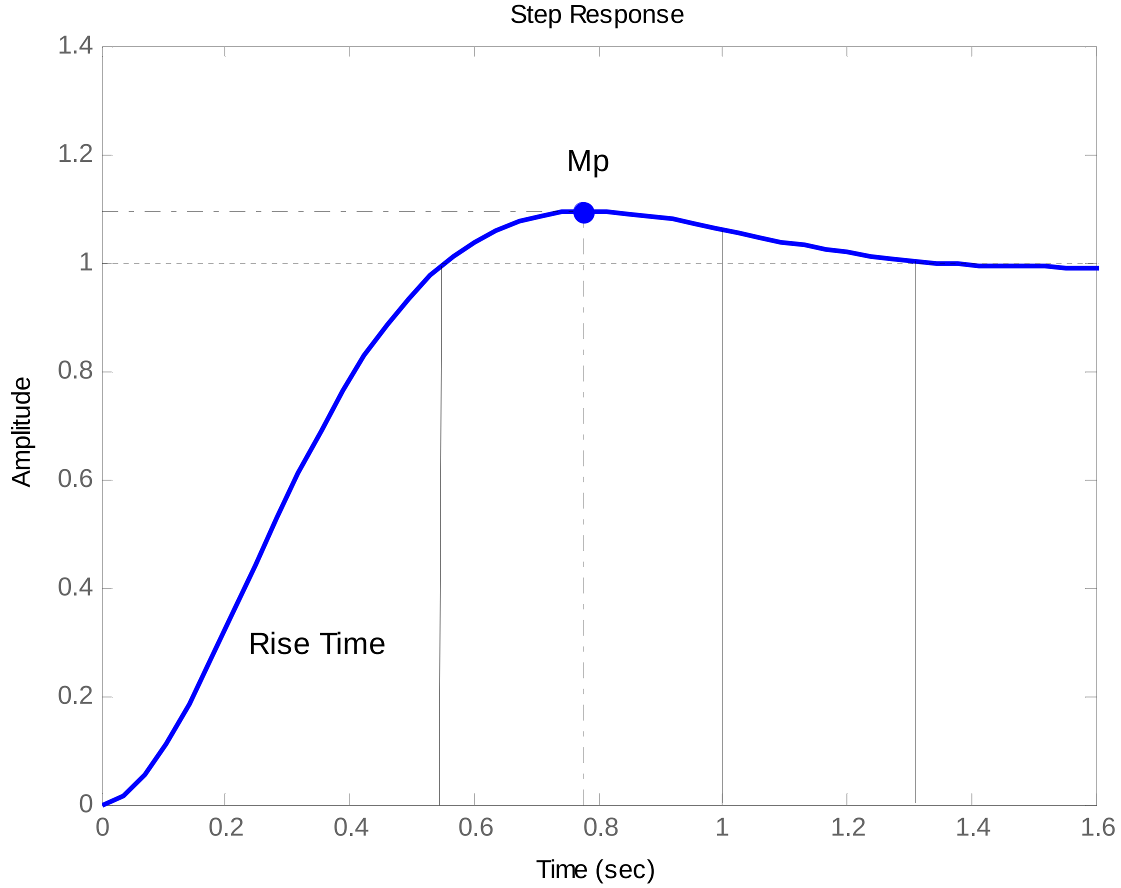
Example -3

Maximum Overshoot

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

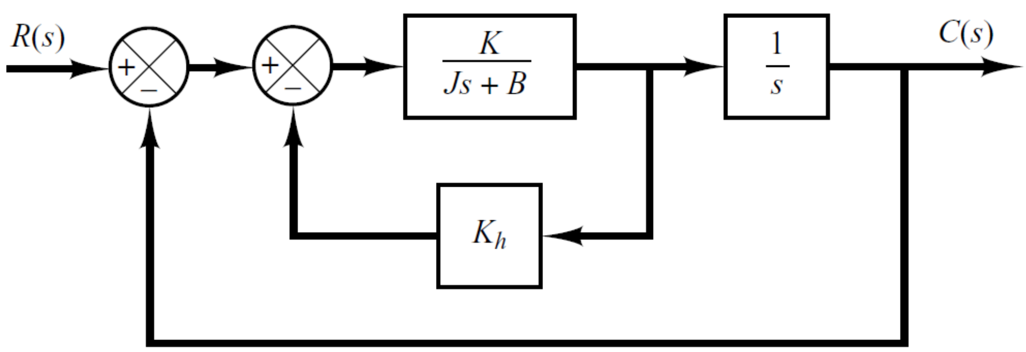
$$M_p = e^{-\frac{3.141 \times 0.6}{\sqrt{1-0.6^2}}} \times 100$$

Example –3

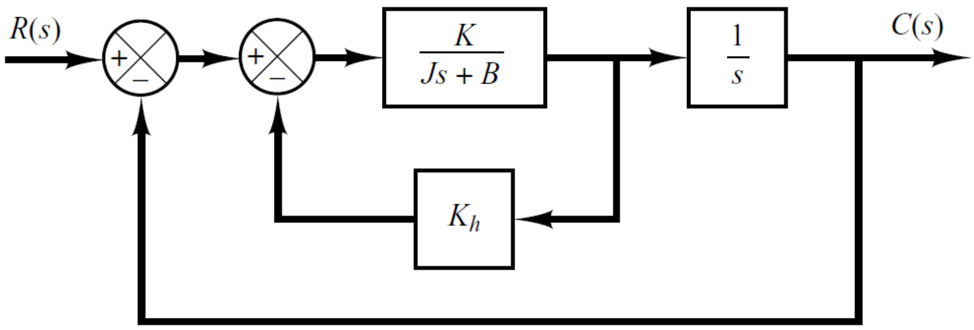


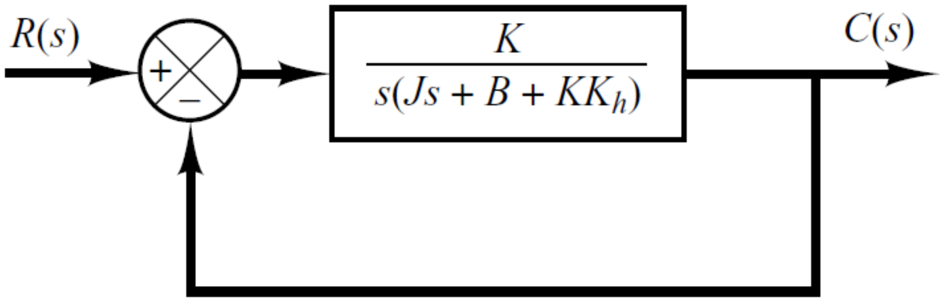
Example –4

- For the system shown in Figure-(a), determine the values of gain K and velocity-feedback constant K_h so that the maximum overshoot in the unit-step response is 0.2 and the peak time is 1 sec. With these values of K and K_h, obtain the rise time and settling time. Assume that $J=1$ kg-m^2 and $B=1$ N-m/rad/sec.



Example –4





$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + (B + KK_h)s + K}$$

Example –4

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + (B + KK_h)s + K}$$

Since $J = 1\ kgm^2$ and $B = 1$ Nm/rad/sec

$$\frac{C(s)}{R(s)} = \frac{K}{s^2 + (1 + KK_h)s + K}$$

- Comparing above T.F with general 2nd order T.F

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n = \sqrt{K} \qquad \zeta = \frac{(1 + KK_h)}{2\sqrt{K}}$$

Example –4

$$\omega_n = \sqrt{K} \qquad \zeta = \frac{(1+KK_h)}{2\sqrt{K}}$$

- Maximum overshoot is 0.2.

$$M_p = e^{-(\zeta/\sqrt{1-\zeta^2})\pi}$$

$$e^{-(\zeta/\sqrt{1-\zeta^2})\pi} = 0.2$$

$$\ln(e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}) = \ln(0.2)$$

$$\frac{\zeta\pi}{\sqrt{1-\zeta^2}} = 1.61$$

$$\zeta = 0.456$$

- The peak time is 1 sec

$$t_p = \frac{\pi}{\omega_d}$$

$$1 = \frac{3.141}{\omega_n\sqrt{1-\zeta^2}}$$

$$\omega_n = \frac{3.141}{\sqrt{1-0.456^2}}$$

$$\omega_n = 3.53$$

Example –4

$$\zeta = 0.456 \qquad \omega_n = 3.53$$

$$\omega_n = \sqrt{K}$$

$$3.53 = \sqrt{K}$$

$$3.53^2 = K$$

$$K = 12.5$$

$$\zeta = \frac{(1+KK_h)}{2\sqrt{K}}$$

$$0.456 \times 2\sqrt{12.5} = (1+12.5K_h)$$

$$K_h = 0.178$$

Example –4

$\zeta = 0.456$

$\omega_n = 3.53$

$$t_r = \frac{\pi - \theta}{\omega_n \sqrt{1-\zeta^2}}$$

$t_r = 0.65s$

$$t_s = \frac{4}{\zeta \omega_n}$$

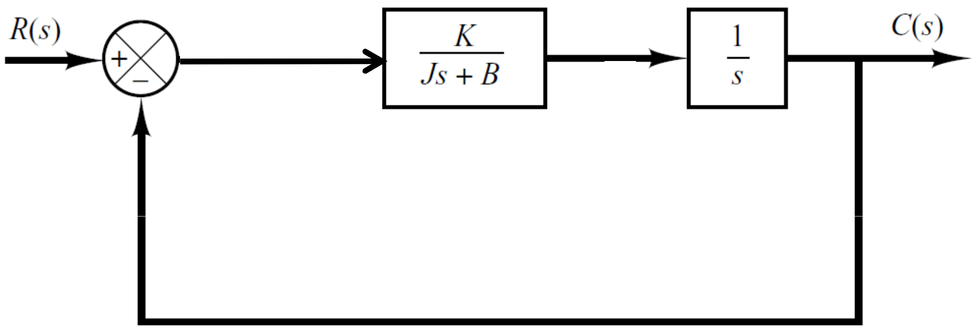
$t_s = 2.48s$

$$t_s = \frac{3}{\zeta \omega_n}$$

$t_s = 1.86s$

Example –4

- Repeat part (a) without the velocity feedback.
- What is your observations ?



Further Reading

Time Domain Specifications (Rise Time)

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

Put $t = t_r$ in above equation

$$c(t_r) = 1 - e^{-\zeta\omega_n t_r}\left[\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r\right]$$

Where $c(t_r) = 1$

$$0 = -e^{-\zeta\omega_n t_r}\left[\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r\right]$$

$-e^{-\zeta\omega_n t_r} \neq 0$

$$0 = \left[\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r\right]$$

Time Domain Specifications (Rise Time)

$$\left[\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r\right] = 0$$

above equation can be re-writen as

$$\sin\omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta}\cos\omega_d t_r$$

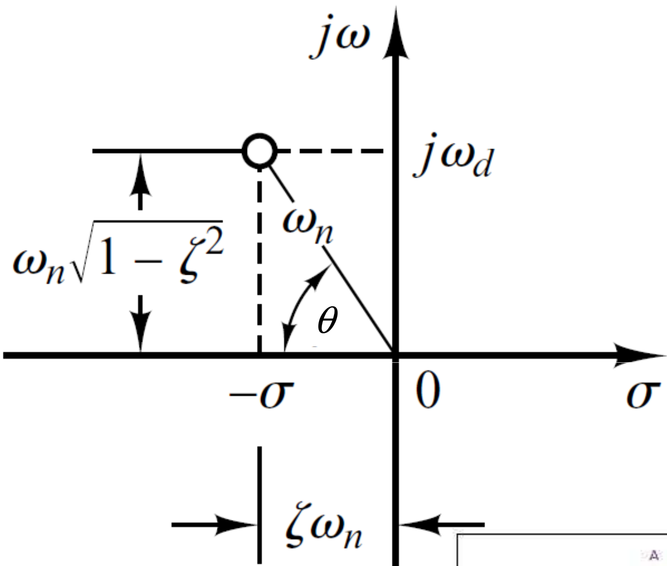
$$\tan\omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\omega_d t_r = \tan^{-1}\left(-\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

Time Domain Specifications (Rise Time)

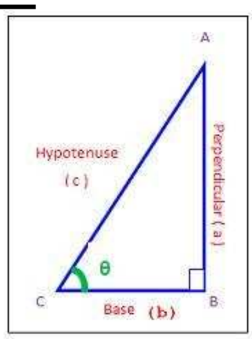
$$\omega_d t_r = \tan^{-1}\left(-\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(-\frac{\omega_n\sqrt{1-\zeta^2}}{\omega_n\zeta}\right)$$



$$t_r = \frac{\pi-\theta}{\omega_d}$$

$$\theta = \tan^{-1}\frac{a}{b}$$



Time Domain Specifications (Peak Time)

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

- In order to find peak time let us differentiate above equation **w.r.t** ***t***.

$$\frac{dc(t)}{dt} = \zeta\omega_n e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right] - e^{-\zeta\omega_n t}\left[-\omega_d \sin\omega_d t + \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos\omega_d t\right]$$

$$0 = e^{-\zeta\omega_n t}\left[\zeta\omega_n \cos\omega_d t + \frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}\sin\omega_d t + \omega_d \sin\omega_d t - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos\omega_d t\right]$$

$$0 = e^{-\zeta\omega_n t}\left[\zeta\cancel{\omega_n} \cos\omega_d t + \frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}\sin\omega_d t + \omega_d \sin\omega_d t - \frac{\zeta\omega_n\cancel{\sqrt{1-\zeta^2}}}{\cancel{\sqrt{1-\zeta^2}}}\cos\omega_d t\right]$$

Time Domain Specifications (Peak Time)

$$0 = e^{-\zeta\omega_n t}\left[\zeta\cancel{\omega_n} \cos\omega_d t + \frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}\sin\omega_d t + \omega_d \sin\omega_d t - \frac{\zeta\omega_n\cancel{\sqrt{1-\zeta^2}}}{\cancel{\sqrt{1-\zeta^2}}}\cos\omega_d t\right]$$

$$e^{-\zeta\omega_n t}\left[\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}\sin\omega_d t + \omega_d \sin\omega_d t\right] = 0$$

$$e^{-\zeta\omega_n t} \neq 0 \qquad \left[\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}\sin\omega_d t + \omega_d \sin\omega_d t\right] = 0$$

$$\sin\omega_d t\left[\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}} + \omega_d\right] = 0$$

Time Domain Specifications (Peak Time)

$$\sin\omega_d t\left[\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}+\omega_d\right]=0$$

$$\left[\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}+\omega_d\right]\neq 0 \qquad \sin\omega_d t = 0$$

$$\omega_d t = \sin^{-1} 0$$

$$t = \frac{0,\ \pi,\ 2\pi,\ \cdots}{\omega_d}$$

- Since for underdamped stable systems first peak is maximum peak therefore,

$$t_p = \frac{\pi}{\omega_d}$$

Time Domain Specifications (Maximum Overshoot)

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

$$c(t_p) = 1 - e^{-\zeta\omega_n t_p}\left[\cos\omega_d t_p + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_p\right]$$

$$c(\infty) = 1$$

$$M_p = \left[\not{1} - e^{-\zeta\omega_n t_p}\left(\cos\omega_d t_p + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_p\right) - \not{1}\right]\times 100$$

Put $t_p = \dfrac{\pi}{\omega_d}$ in above equation

$$M_p = \left[-e^{-\zeta\omega_n\frac{\pi}{\omega_d}}\left(\cos\omega_d\frac{\pi}{\omega_d} + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d\frac{\pi}{\omega_d}\right)\right]\times 100$$

Time Domain Specifications (Maximum Overshoot)

$$M_p = \left[-e^{-\zeta\omega_n \frac{\pi}{\omega_d}} \left(\cos \omega_d \frac{\pi}{\omega_d} + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d \frac{\pi}{\omega_d} \right) \right] \times 100$$

Put $\omega_d = \omega_n \sqrt{1-\zeta^2}$ in above equation

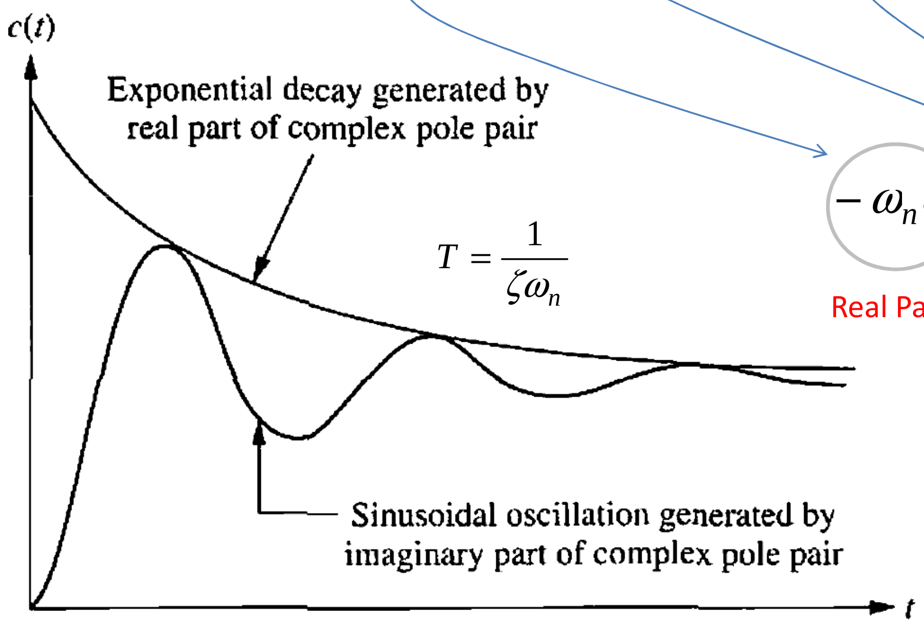
$$M_p = \left[-e^{-\zeta\omega_n \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}} \left(\cos \pi + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \pi \right) \right] \times 100$$

$$M_p = \left[-e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} (-1+0) \right] \times 100$$

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

Time Domain Specifications (Settling Time)

$$c(t) = 1 - e^{-\zeta\omega_n t} \left[\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t \right]$$



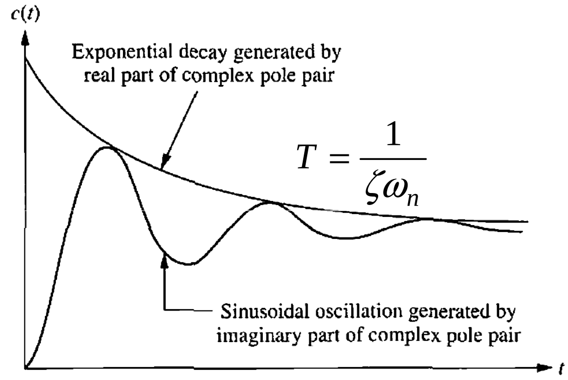
$$-\omega_n \zeta \pm \omega_n \sqrt{\zeta^2 - 1}$$

Real Part   Imaginary Part

Time Domain Specifications (Settling Time)

- Settling time (2%) criterion
  - Time consumed in exponential decay up to 98% of the input.

$$t_s = 4T = \frac{4}{\zeta\omega_n}$$



- Settling time (5%) criterion
  - Time consumed in exponential decay up to 95% of the input.

$$t_s = 3T = \frac{3}{\zeta\omega_n}$$

Step Response of critically damped System ($\zeta = 1$)

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s+\omega_n)^2} \xrightarrow{\text{Step Response}} C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2}$$

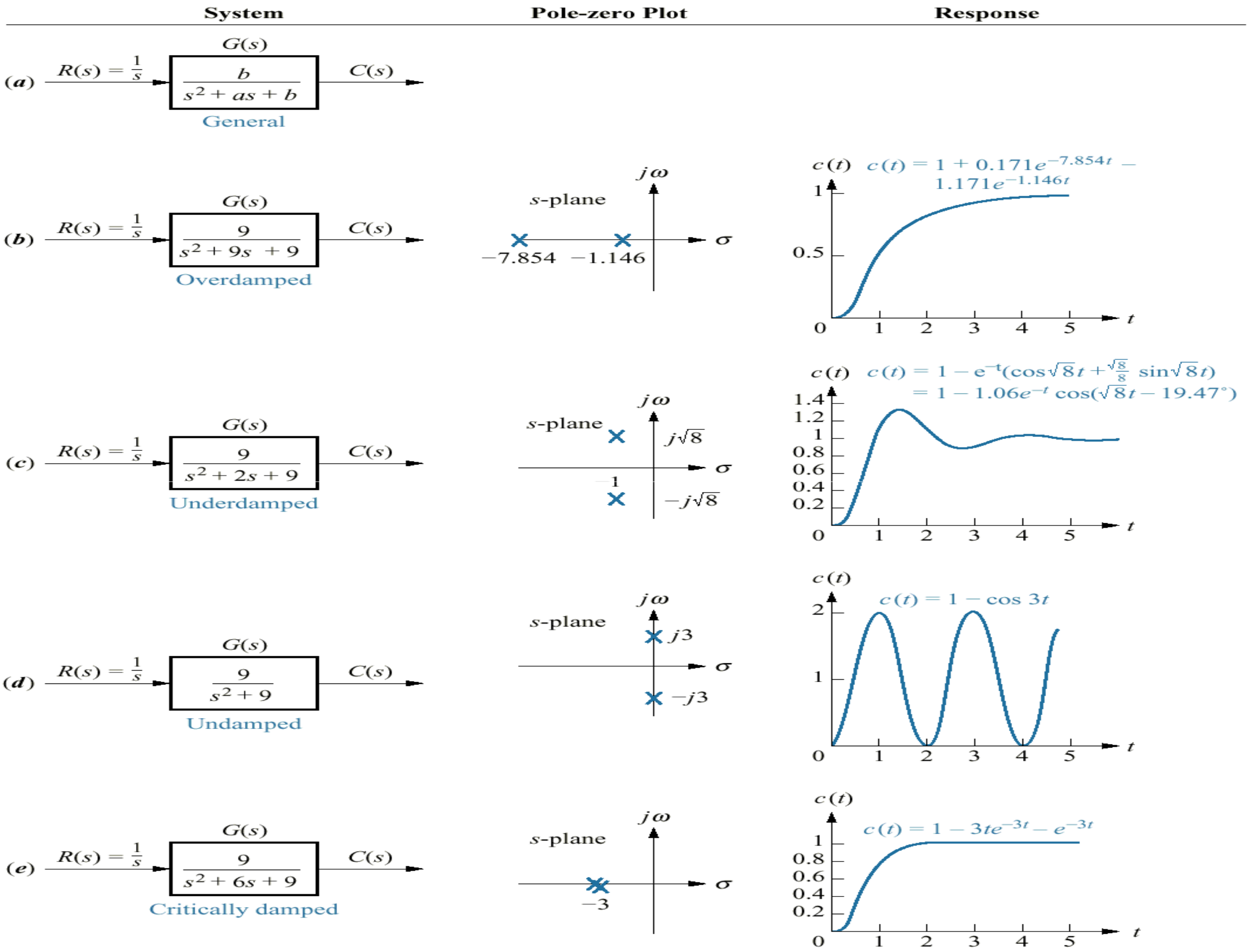
- The partial fraction expansion of above equation is given as

$$\frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{s+\omega_n} + \frac{C}{(s+\omega_n)^2}$$

$$C(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

$$c(t) = 1 - e^{-\omega_n t} - \omega_n e^{-\omega_n t} t$$

$$c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)$$



System
Pole-zero Plot
Response
(a)
R(s) = 1/s
G(s)
b / (s² + as + b)
C(s)
General
(b)
R(s) = 1/s
G(s)
9 / (s² + 9s + 9)
C(s)
Overdamped
jω
s-plane
σ
−7.854
−1.146
c(t)
c(t) = 1 + 0.171e^(−7.854t) − 1.171e^(−1.146t)
1
0.5
0
1
2
3
4
5
t
(c)
R(s) = 1/s
G(s)
9 / (s² + 2s + 9)
C(s)
Underdamped
jω
s-plane
j√8
σ
−1
−j√8
c(t)
c(t) = 1 − e^(−t)(cos√8t + (√8/8) sin√8t)
= 1 − 1.06e^(−t) cos(√8t − 19.47°)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
1
2
3
4
5
t
(d)
R(s) = 1/s
G(s)
9 / (s² + 9)
C(s)
Undamped
jω
s-plane
j3
σ
−j3
c(t)
c(t) = 1 − cos 3t
2
1
0
1
2
3
4
5
t
(e)
R(s) = 1/s
G(s)
9 / (s² + 6s + 9)
C(s)
Critically damped
jω
s-plane
σ
−3
c(t)
c(t) = 1 − 3te^(−3t) − e^(−3t)
1
0.8
0.6
0.4
0.2
0
1
2
3
4
5
t