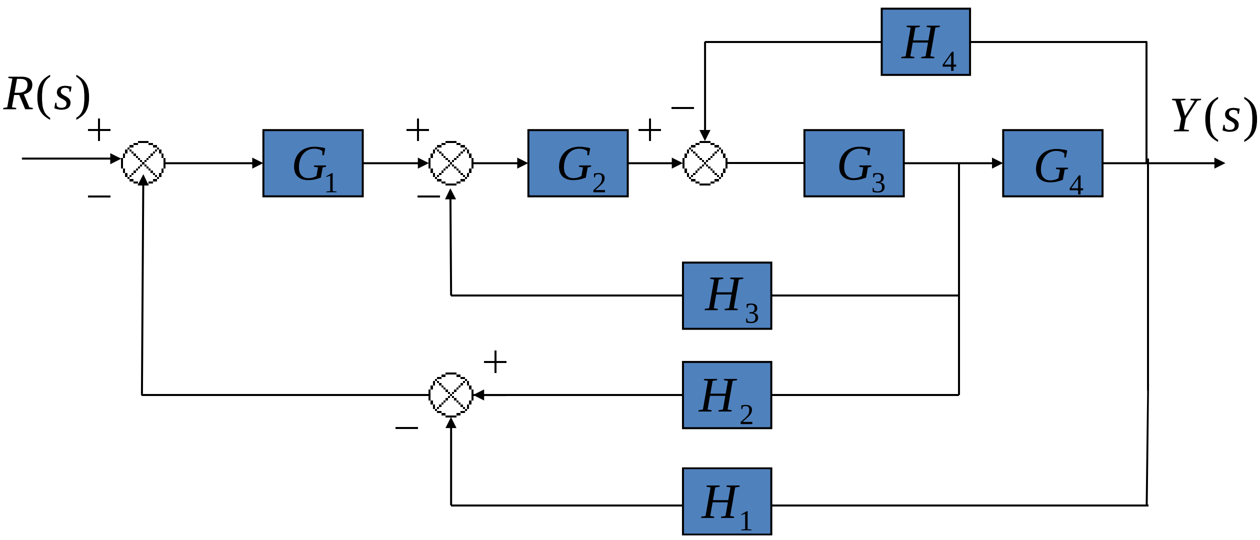
Automatic Control Theory

CSE 322

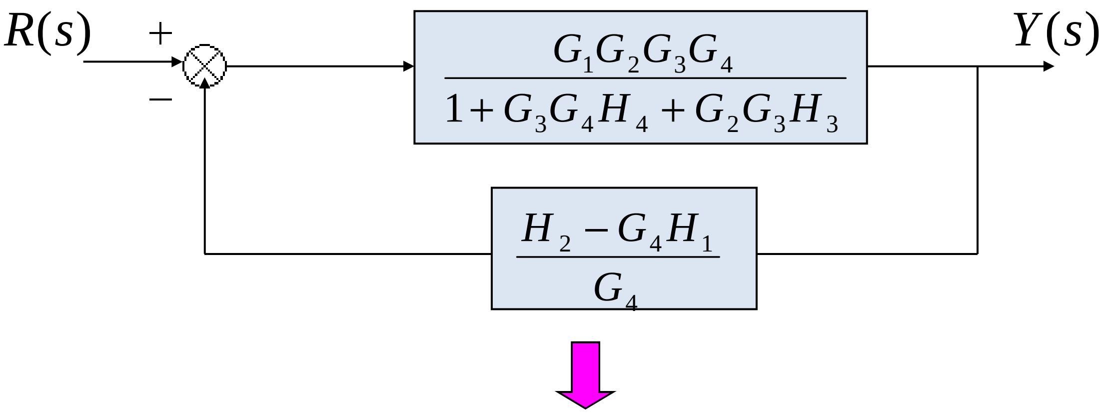
Lec. 5
Signal Flow Graph

Quiz (10 mins)

Find the transfer function of the following block diagrams



3. Eliminate loop II & IIII



$$\frac{Y(s)}{R(s)} = \frac{G_1G_2G_3G_4}{1+G_2G_3H_3+G_3G_4H_4+G_1G_2G_3H_2-G_1G_2G_3G_4H_1}$$

Outline

- Introduction to Signal Flow Graphs
  - Definitions
  - Terminologies
  - Examples
- Mason's Gain Formula
  - Examples
- Signal Flow Graph from Block Diagrams
- Design Examples

Introduction

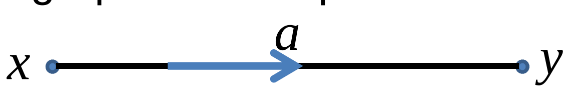
- Alternative method to block diagram representation, developed by Samuel Jefferson Mason.
- A signal-flow graph consists of a network in which nodes are connected by directed branches.
- It depicts the flow of signals from one point of a system to another and gives the relationships among the signals.

Fundamentals of Signal Flow Graphs

- Consider a simple equation below and draw its signal flow graph:

$$y = ax$$

- The signal flow graph of the equation is shown below;

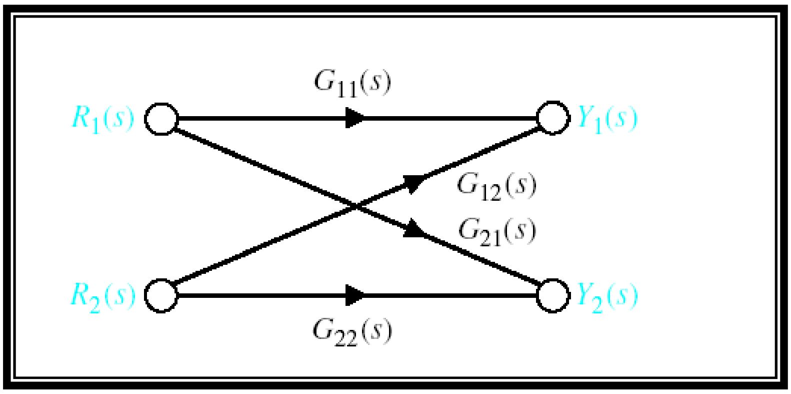


- Every variable in a signal flow graph is designed by a **Node**.
- Every transmission function in a signal flow graph is designed by a **Branch**.
- Branches are always **unidirectional**.
- The arrow in the branch denotes the **direction** of the signal flow.

Signal-Flow Graph Models

$$Y_1(s) = G_{11}(s) \cdot R_1(s) + G_{12}(s) \cdot R_2(s)$$

$$Y_2(s) = G_{21}(s) \cdot R_1(s) + G_{22}(s) \cdot R_2(s)$$

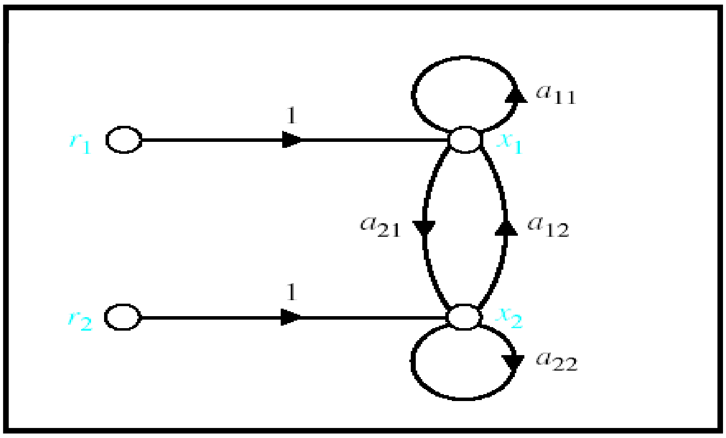


Signal-Flow Graph Models

r_1 and r_2 are inputs and ***x_1*** and ***x_2*** are outputs

$$a_{11} \cdot x_1 + a_{12} \cdot x_2 + r_1 = x_1$$

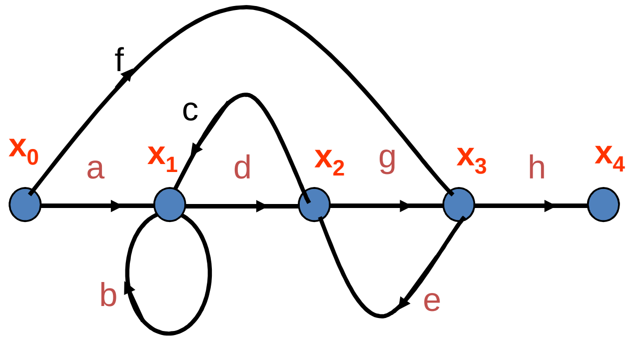
$$a_{21} \cdot x_1 + a_{22} \cdot x_2 + r_2 = x_2$$



Signal-Flow Graph Models

X_o is input and X_4 is output

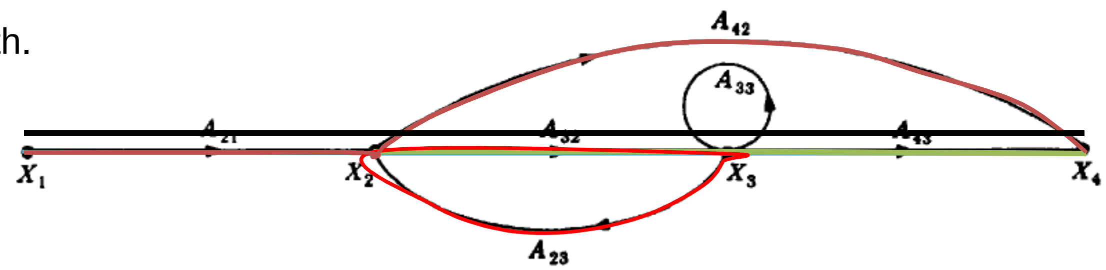
$$x_1 = ax_0 + bx_1 + cx_2$$
$$x_2 = dx_1 + ex_3$$
$$x_3 = fx_0 + gx_2$$
$$x_4 = hx_3$$



Terminologies

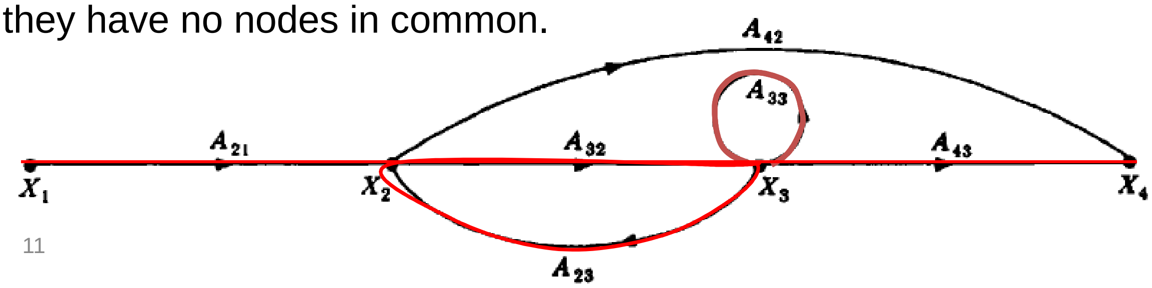
- An **input node** or source contain only the outgoing branches. i.e., X_1
- An **output node** or sink contain only the incoming branches. i.e., X_4
- A **path** is a continuous, unidirectional succession of branches along which no node is passed more than ones. i.e.,

  X_1 to X_2 to X_3 to X_4 *X_1 to X_2 to X_4* *X_2 to X_3 to X_4*
- A **forward path** is a path from the input node to the output node. i.e., *X_1 to X_2 to X_3 to X_4*, and *X_1 to X_2 to X_4*, are forward paths.
- A **feedback path** or feedback loop is a path which originates and terminates on the same node. i.e.; *X_2 to X_3* **and back to** *X_2* is a feedback path.

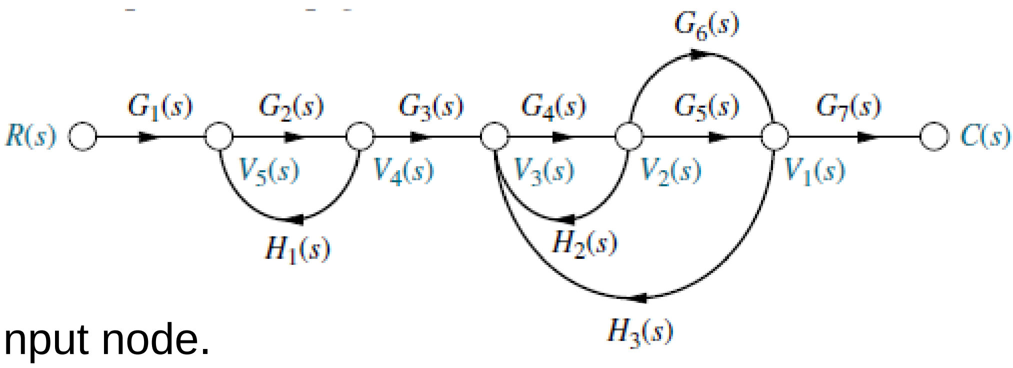


Terminologies

- A **self-loop** is a feedback loop consisting of a single branch. i.e.; A_{33} is a self loop.
- The **gain** of a branch is the transmission function of that branch.
- The **path gain** is the product of branch gains encountered in traversing a path. i.e. the gain of forwards path X_1 to X_2 to X_3 to X_4 is $A_{21}A_{32}A_{43}$
- The **loop gain** is the product of the branch gains of the loop. i.e., the loop gain of the feedback loop from X_2 to X_3 and back to X_2 is $A_{32}A_{23}$.
- Two loops, paths, or loop and a path are said to be **non-touching** if they have no nodes in common.

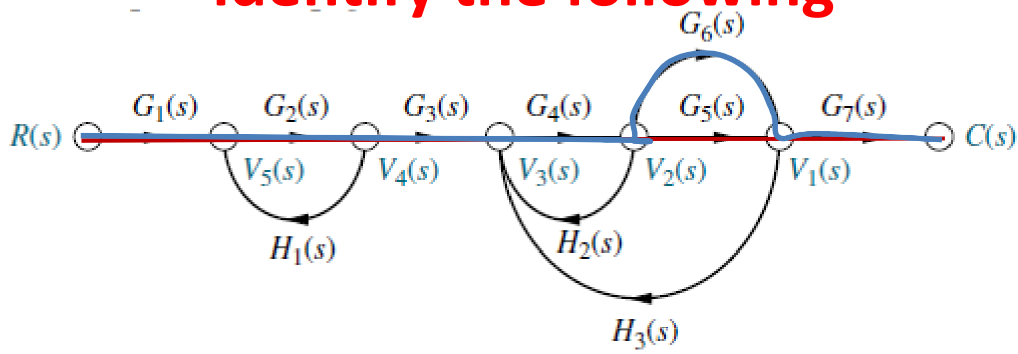


Consider the signal flow graph below and identify the following



a) Input node.
b) Output node.
c) Forward paths.
d) Feedback paths (loops).
e) Determine the loop gains of the feedback loops.
f) Determine the path gains of the forward paths.
g) Non-touching loops

Consider the signal flow graph below and identify the following



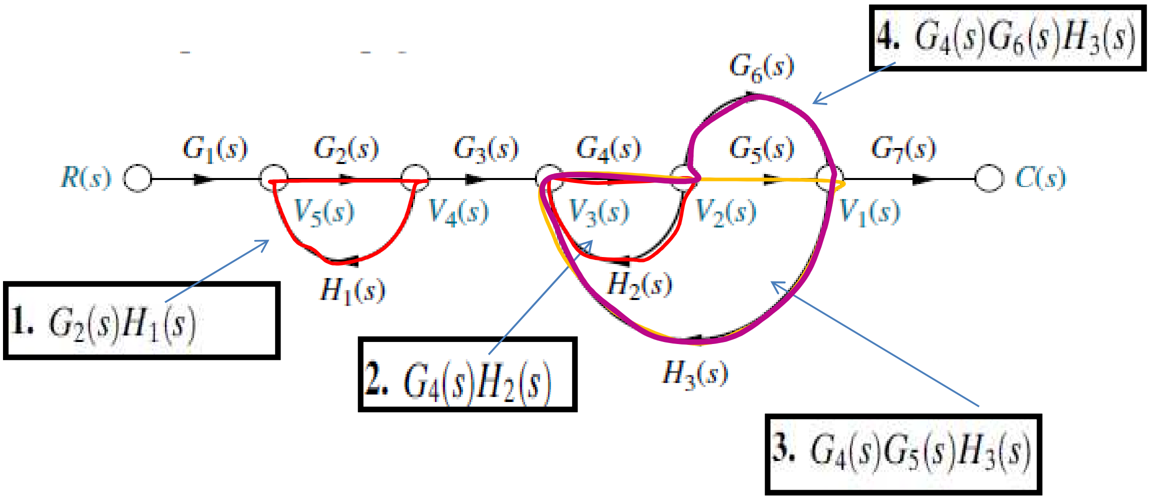
- There are **two** forward path gains;

1. $G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)G_7(s)$

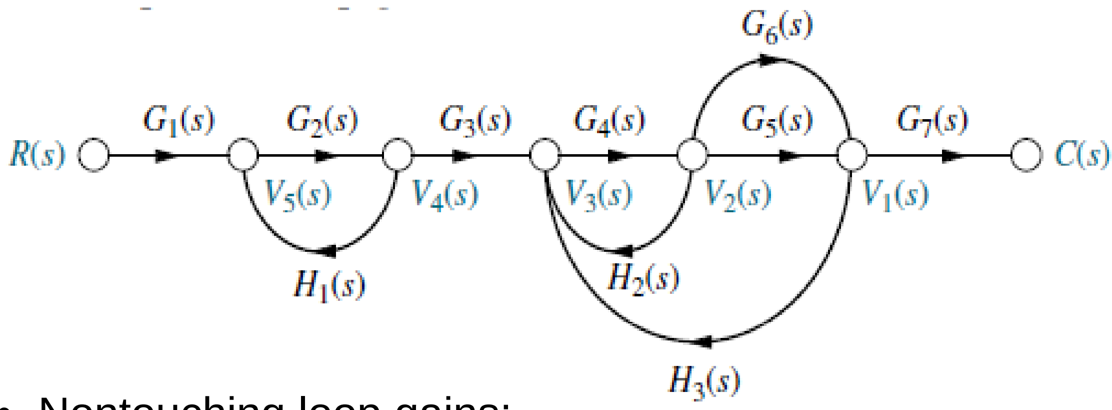
2. $G_1(s)G_2(s)G_3(s)G_4(s)G_6(s)G_7(s)$

Consider the signal flow graph below and identify the following

- There are **four** loops



Consider the signal flow graph below and identify the following



- Nontouching loop gains;

1.	$[G_2(s)H_1(s)][G_4(s)H_2(s)]$
2.	$[G_2(s)H_1(s)][G_4(s)G_5(s)H_3(s)]$
3.	$[G_2(s)H_1(s)][G_4(s)G_6(s)H_3(s)]$

Mason's Rule (Mason, 1953)

- The block diagram reduction technique requires successive application of fundamental relationships in order to arrive at the system transfer function.
- On the other hand, Mason's rule for reducing a signal-flow graph to a single transfer function requires the application of one formula.
- The formula was derived by S. J. Mason when he related the signal-flow graph to the simultaneous equations that can be written from the graph.

Mason's Rule:

- The transfer function, *C(s)/R(s),* of a system represented by a signal-flow graph is;

$$\frac{C(s)}{R(s)} = \frac{\sum_{i=1}^{n} P_i \Delta_i}{\Delta}$$

Where:

n = number of forward paths.
P_i = the **i** th forward-path gain.
Δ = Determinant of the system
Δ_i = Determinant of the **i**th forward path

Δ is called the signal flow graph determinant or **characteristic function**. Since Δ=0 is the system characteristic equation.

Mason's Rule:

$$\frac{C(s)}{R(s)} = \frac{\sum_{i=1}^{n} P_i \Delta_i}{\Delta}$$

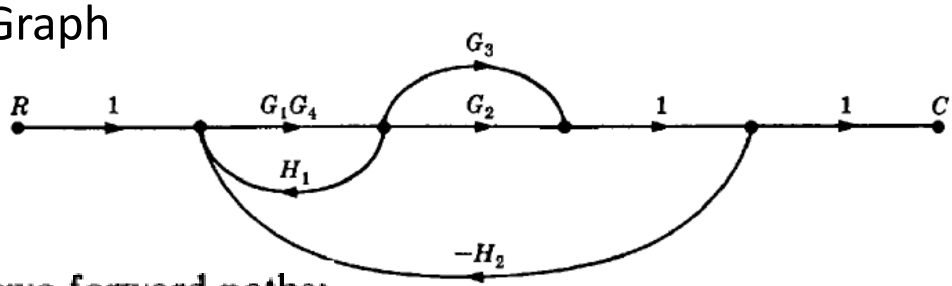
Δ = 1- (sum of all individual loop gains) + (sum of the products of the gains of all possible two loops that do not touch each other) – (sum of the products of the gains of all possible three loops that do not touch each other) + … and so forth with sums of higher number of non-touching loop gains

Δ_i = value of Δ for the part of the block diagram that does not touch the i-th forward path (Δ_i = 1 if there are **no non-touching** loops to the i-th path.)

Systematic approach

1. Calculate forward path gain P_i for each forward path i.
2. Calculate all loop transfer functions
3. Consider non-touching loops 2 at a time
4. Consider non-touching loops 3 at a time
5. etc
6. Calculate Δ from steps 2,3,4 and 5
7. Calculate Δ_i as portion of Δ not touching forward path i

Example-1: Apply Mason's Rule to calculate the transfer function of the system represented by following Signal Flow Graph



There are two forward paths:

$P_1 = G_1G_2G_4 \qquad P_2 = G_1G_3G_4$

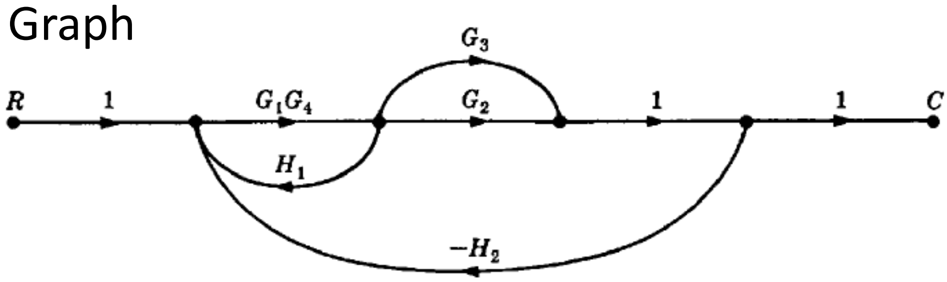
Therefore,

$$\frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

There are <u>three</u> feedback loops

$$L_1 = G_1G_4H_1, \quad L_2 = -G_1G_2G_4H_2, \quad L_3 = -G_1G_3G_4H_2$$

Example-1: Apply Mason's Rule to calculate the transfer function of the system represented by following Signal Flow Graph



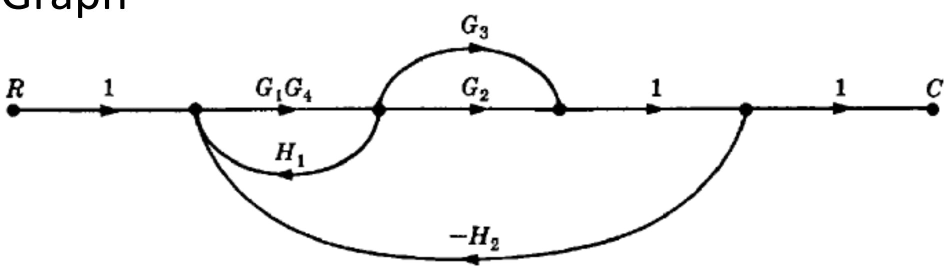
There are no non-touching loops, therefore

Δ = 1- (sum of all individual loop gains)

$$\Delta = 1 - (L_1 + L_2 + L_3)$$

$$\Delta = 1 - (G_1G_4H_1 - G_1G_2G_4H_2 - G_1G_3G_4H_2)$$

Example-1: Apply Mason's Rule to calculate the transfer function of the system represented by following Signal Flow Graph



Eliminate forward path-1

Δ_1 = 1- (sum of all individual loop gains)+...

$\Delta_1 = 1$

Eliminate forward path-2

Δ_2 = 1- (sum of all individual loop gains)+...

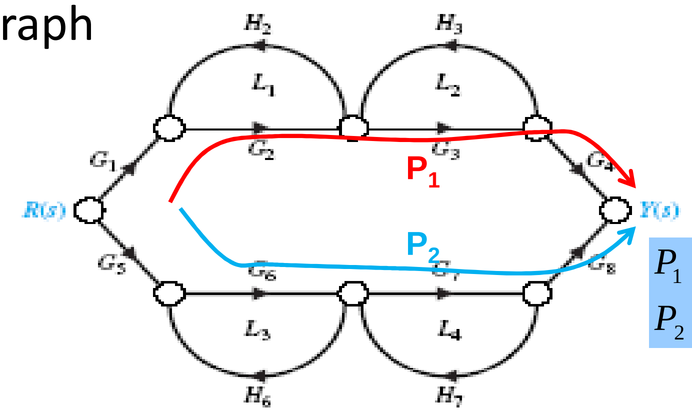
$\Delta_2 = 1$

Example-1: Continue

$$\frac{C}{R}=\frac{P_1\Delta_1+P_2\Delta_2}{\Delta}=\frac{G_1G_2G_4+G_1G_3G_4}{1-G_1G_4H_1+G_1G_2G_4H_2+G_1G_3G_4H_2}$$

$$=\frac{G_1G_4(G_2+G_3)}{1-G_1G_4H_1+G_1G_2G_4H_2+G_1G_3G_4H_2}$$

Example-2: Apply Mason's Rule to calculate the transfer function of the system represented by following Signal Flow Graph



$P_1=G_1G_2G_3G_4$ (path 1) and
$P_2=G_5G_6G_7G_8$ (path 2)

$L_1=G_2H_2,\quad L_2=H_3G_3,\quad L_3=G_6H_6,\quad L_4=G_7H_7$

1. Calculate forward path gains for each forward path.

2. Calculate all loop gains.

Example -2: continue

3. Consider two non-touching loops.

L_1L_3 L_1L_4

L_2L_4 L_2L_3

4. Consider three non-touching loops.

   None.

5. Calculate Δ from steps 2,3,4.

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4) + (L_1L_3 + L_1L_4 + L_2L_3 + L_2L_4)$$

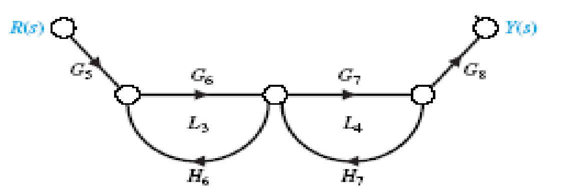
$$\Delta = 1 - (G_2H_2 + H_3G_3 + G_6H_6 + G_7H_7) + (G_2H_2G_6H_6 + G_2H_2G_7H_7 + H_3G_3G_6H_6 + H_3G_3G_7H_7)$$

Example -2: continue

Eliminate forward path-1

$$\Delta_1 = 1 - (L_3 + L_4)$$

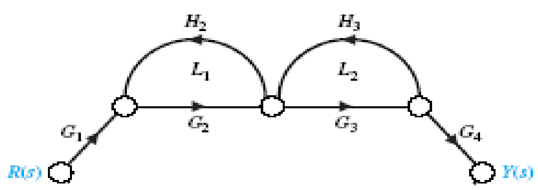
$$\Delta_1 = 1 - (G_6H_6 + G_7H_7)$$



Eliminate forward path-2

$$\Delta_2 = 1 - (L_1 + L_2)$$

$$\Delta_2 = 1 - (G_2H_2 + G_3H_3)$$



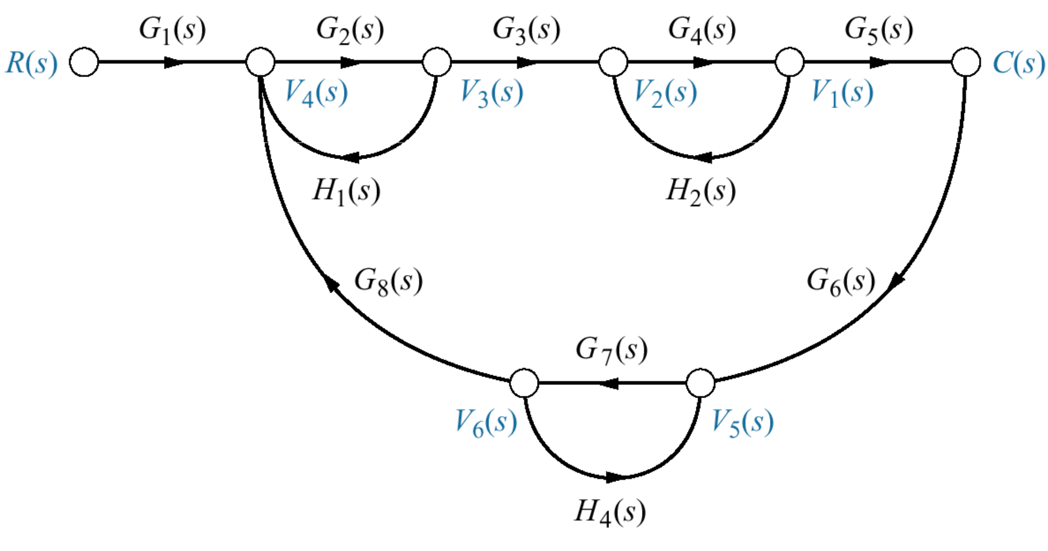
Example -2: continue

$$\frac{Y(s)}{R(s)} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$$\frac{Y(s)}{R(s)} = \frac{G_1G_2G_3G_4\left[1-\left(G_6H_6+G_7H_7\right)\right]+G_5G_6G_7G_8\left[1-\left(G_2H_2+G_3H_3\right)\right]}{1-\left(G_2H_2+H_3G_3+G_6H_6+G_7H_7\ \right)+\left(G_2H_2G_6H_6+G_2H_2G_7H_7+H_3G_3G_6H_6+H_3G_3G_7H_7\right)}$$

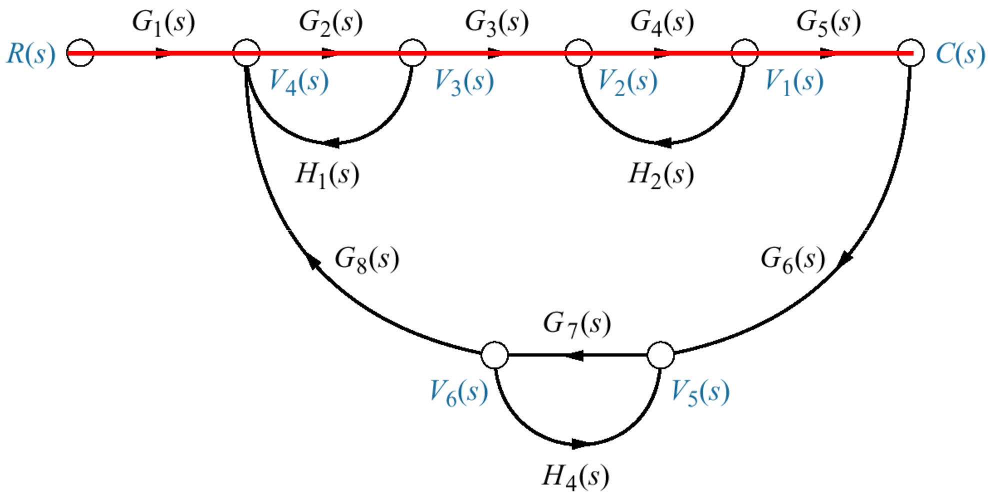
Example#3

- Find the transfer function, C(s)/R(s), for the signal-flow graph in figure below.



Example#3

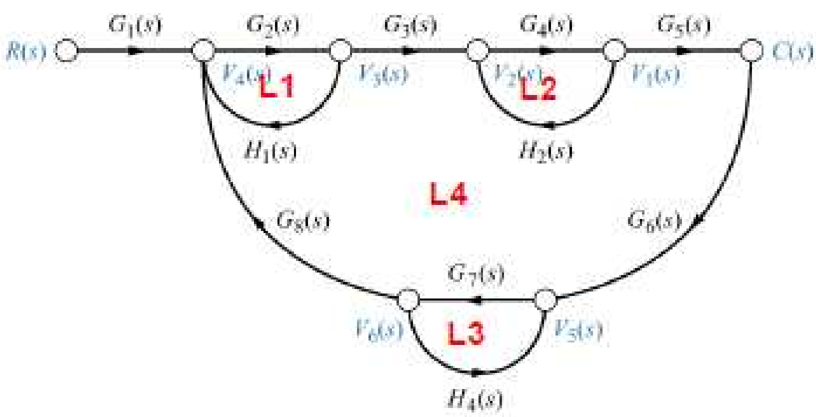
- There is only one forward Path.



$$P_1 = G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)$$

Example#3

- There are four feedback loops.



L1. $G_2(s)H_1(s)$

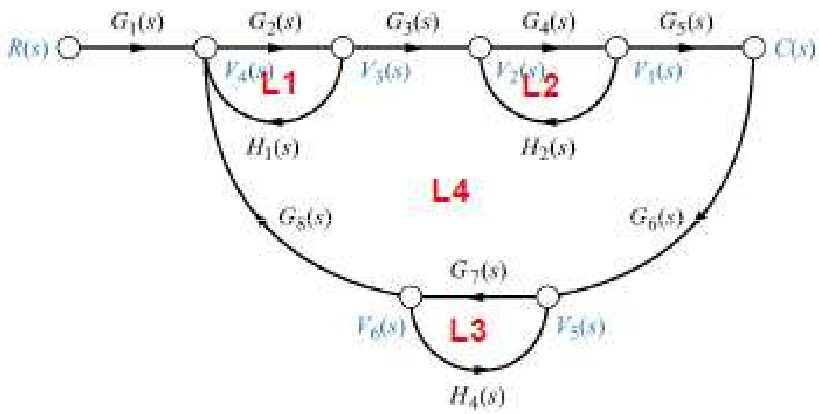
L2. $G_4(s)H_2(s)$

L3. $G_7(s)H_4(s)$

L4. $G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)$

Example#3

- Non-touching loops taken two at a time.

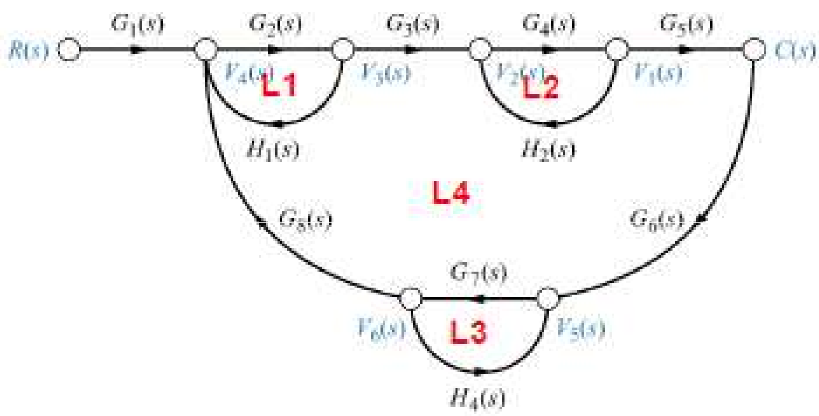


L1 and L2: $G_2(s)H_1(s)G_4(s)H_2(s)$ L2 and L3: $G_4(s)H_2(s)G_7(s)H_4(s)$

L1 and L3: $G_2(s)H_1(s)G_7(s)H_4(s)$

Example#3

- Non-touching loops taken three at a time.



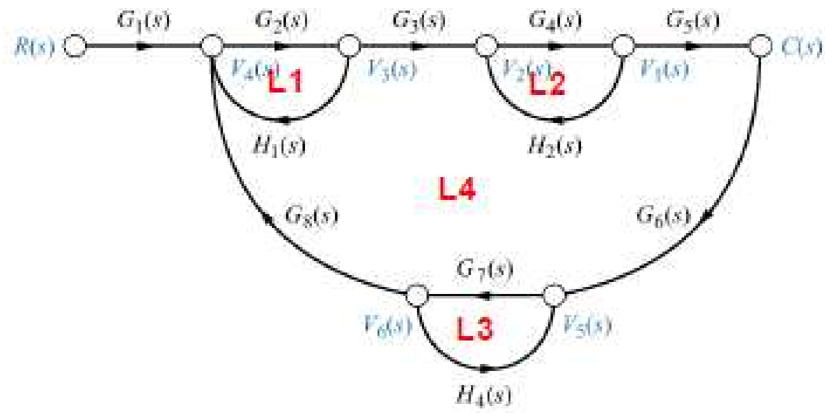
L1, L2, L3: $G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)$

Example#3

$$\Delta = 1 - [G_2(s)H_1(s) + G_4(s)H_2(s)$$
$$+ G_7(s)H_4(s) + G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)]$$
$$+ [G_2(s)H_1(s)G_4(s)H_2(s) + G_2(s)H_1(s)G_7(s)H_4(s)$$
$$+ G_4(s)H_2(s)G_7(s)H_4(s)]$$
$$- [G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)]$$

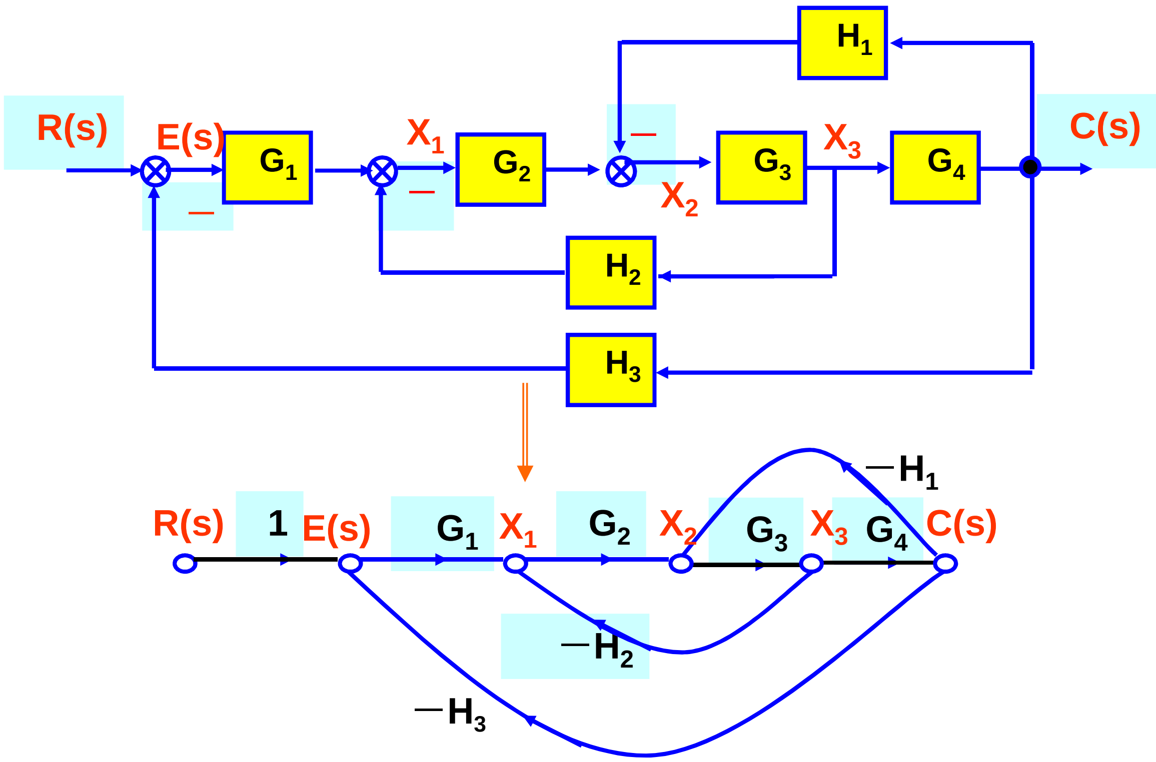
Eliminate forward path-1

$$\Delta_1 = 1 - G_7(s)H_4(s)$$

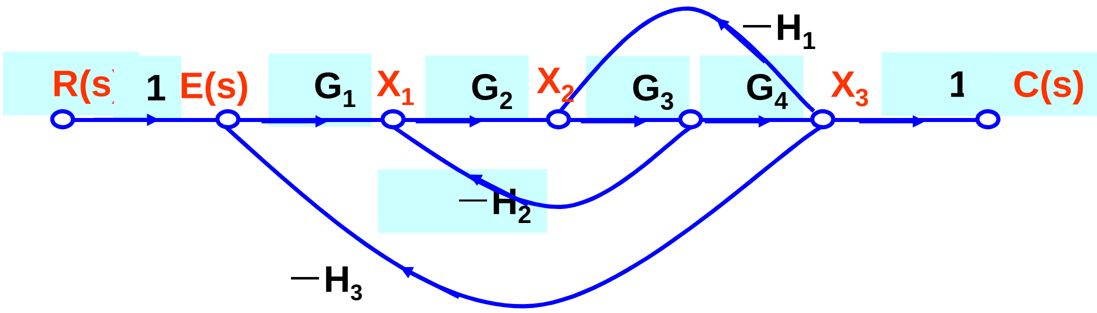


From Block Diagram to Signal-Flow Graph Models

Example 5:



Example 5:



$$\Delta = 1 + (G_1G_2G_3G_4H_3 + G_2G_3H_2 + G_3G_4H_1)$$

$$P_1 = G_1G_2G_3G_4; \qquad \Delta_1 = 1$$

$$G = \frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4}{1 + G_1G_2G_3G_4H_3 + G_2G_3H_2 + G_3G_4H_1}$$