Automatic Control Theory

CSE 322

Lec. 6
Time Domain Analysis
(1st Order Systems)

Introduction

- In time-domain analysis the response of a dynamic system to an input is expressed as a function of time.
- It is possible to compute the time response of a system if the nature of input and the mathematical model of the system are known.
- Usually, the input signals to control systems are not known fully ahead of time.
- For example, in a radar tracking system, the position and the speed of the target to be tracked may vary in a random fashion.
- It is therefore difficult to express the actual input signals mathematically by simple equations.

Standard Test Signals

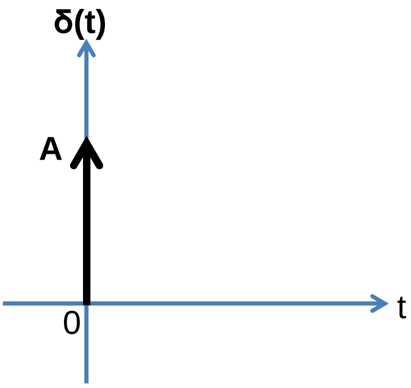
- The characteristics of actual input signals are a sudden shock, a sudden change, a constant velocity, and constant acceleration.
- The dynamic behavior of a system is therefore judged and compared under application of standard test signals – an impulse, a step, a constant velocity, and constant acceleration.
- Another standard signal of great importance is a sinusoidal signal.

Standard Test Signals

- **Impulse signal**
  - The impulse signal imitate the sudden shock characteristic of actual input signal.

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

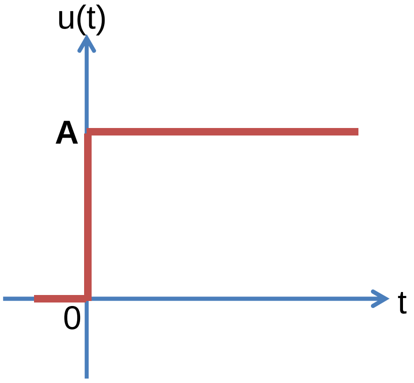
  - If **A=1**, the impulse signal is called unit impulse signal.



Standard Test Signals

- **Step signal**
  - The step signal imitate the sudden change characteristic of actual input signal.

$$u(t)=\begin{cases}A & t\geq 0\\ 0 & t<0\end{cases}$$

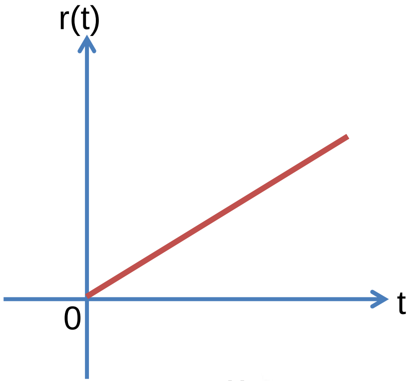


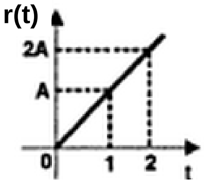
  - If A=1, the step signal is called unit step signal

Standard Test Signals

- **Ramp signal**
  - The ramp signal imitate the constant velocity characteristic of actual input signal.

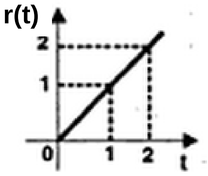
$$r(t)=\begin{cases}At & t\geq 0\\ 0 & t<0\end{cases}$$





ramp signal with slope A

  - If *A=1*, the ramp signal is called unit ramp signal



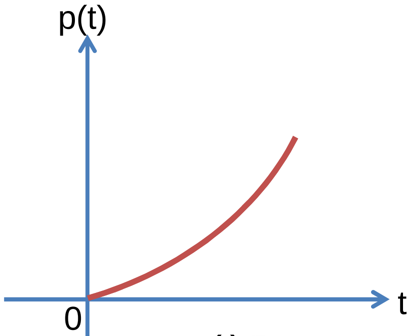
unit ramp signal

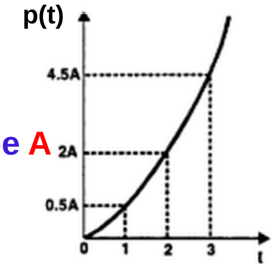
Standard Test Signals

- **Parabolic signal**
  - The parabolic signal imitate the constant acceleration characteristic of actual input signal.

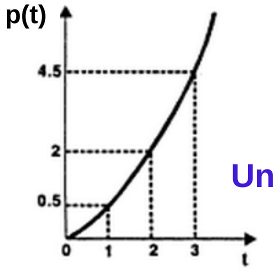
$$p(t) = \begin{cases} \dfrac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

  - If *A=1*, the parabolic signal is called unit parabolic signal.





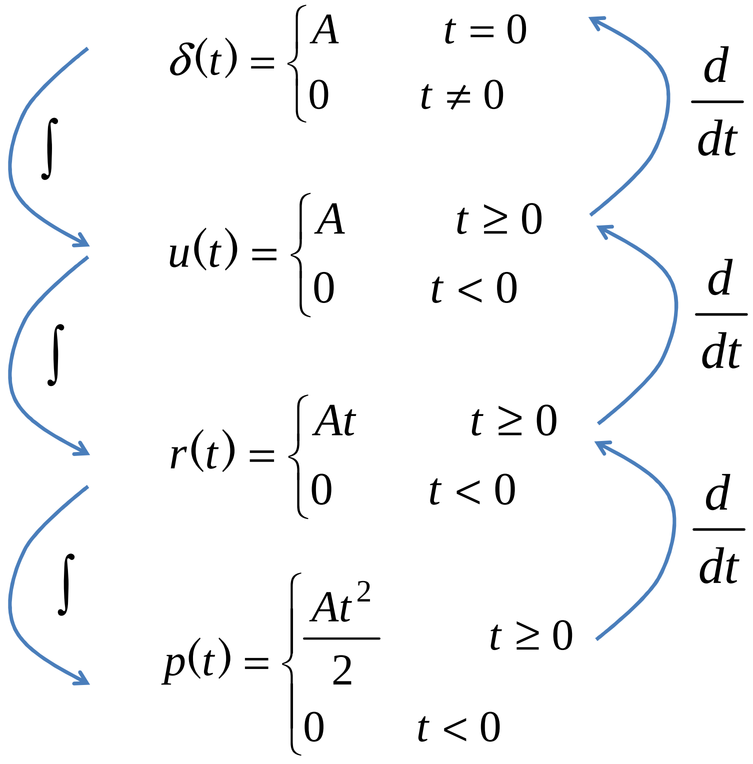
parabolic signal with slope A



Unit parabolic signal

Relation between standard Test Signals

- Impulse
- Step
- Ramp
- Parabolic



$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$p(t) = \begin{cases} \dfrac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Laplace Transform of Test Signals

- Impulse

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$$L\{\delta(t)\} = \delta(s) = A$$

- Step

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{u(t)\} = U(s) = \frac{A}{S}$$

Laplace Transform of Test Signals

- Ramp

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{r(t)\} = R(s) = \frac{A}{s^2}$$

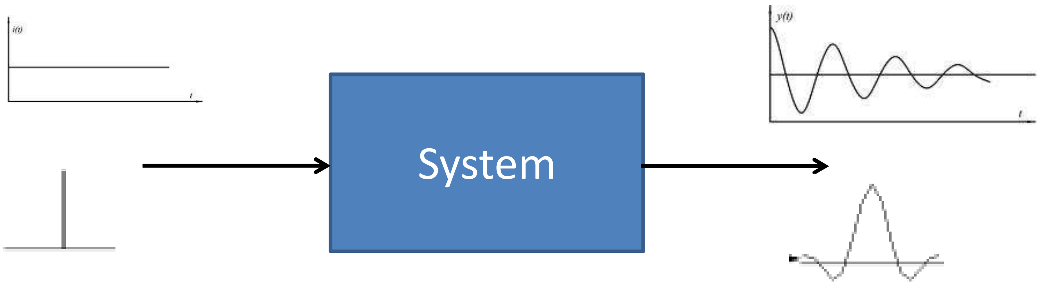
- Parabolic

$$p(t) = \begin{cases} \frac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{p(t)\} = P(s) = \frac{2A}{S^3}$$

Time Response of Control Systems

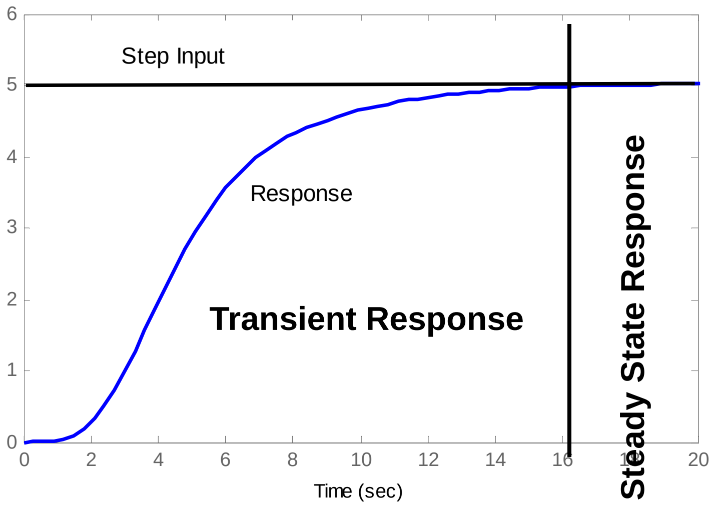
- Time response of a dynamic system response to an input expressed as a function of time.



- The time response of any system has **two** components
  - ➢**Transient response**
  - ➢**Steady-state response.**

Time Response of Control Systems

- When the response of the system is changed form rest or equilibrium it takes some time to settle down.
  - **Transient response** is the response of a system from rest or equilibrium to steady state.
- The response of the system after the transient response is called **steady state** response.



Time Response of Control Systems

- Transient response depend upon the system **poles** only and not on the type of input.
- It is therefore sufficient to analyze the transient response using a step input.
- The steady-state response depends on system dynamics and the input quantity.
- It is then examined using different test signals by final value theorem.

Introduction

- The first order system has only one pole.

$$\frac{C(s)}{R(s)} = \frac{K}{Ts+1}$$

- Where ***K*** is the D.C gain and ***T*** is the time constant of the system.
- Time constant is a measure of how quickly a 1st order system responds to a unit step input.
- D.C Gain of the system is ratio between the input signal and the steady state value of output.

Introduction

- The first order system given below.

$$G(s) = \frac{10}{3s+1}$$

- D.C gain is 10 and time constant is 3 seconds.

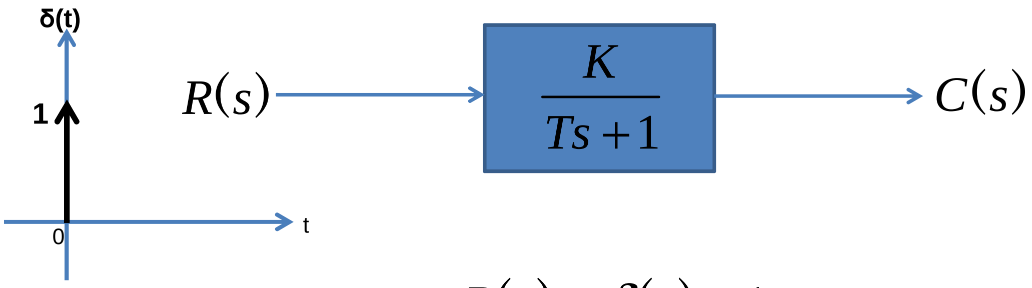
- And for following system

$$G(s) = \frac{3}{s+5} = \frac{3/5}{1/5s+1}$$

- D.C Gain of the system is 3/5 and time constant is 1/5 seconds.

Impulse Response of 1^{st} Order System

- Consider the following 1^{st} order system



$$R(s) = \delta(s) = 1$$

$$C(s) = \frac{K}{Ts+1}$$

Impulse Response of 1st Order System

$$C(s) = \frac{K}{Ts+1}$$

- Re-arrange following equation as

$$C(s) = \frac{K/T}{s+1/T}$$

- In order represent the response of the system in time domain we need to compute inverse Laplace transform of the above equation.

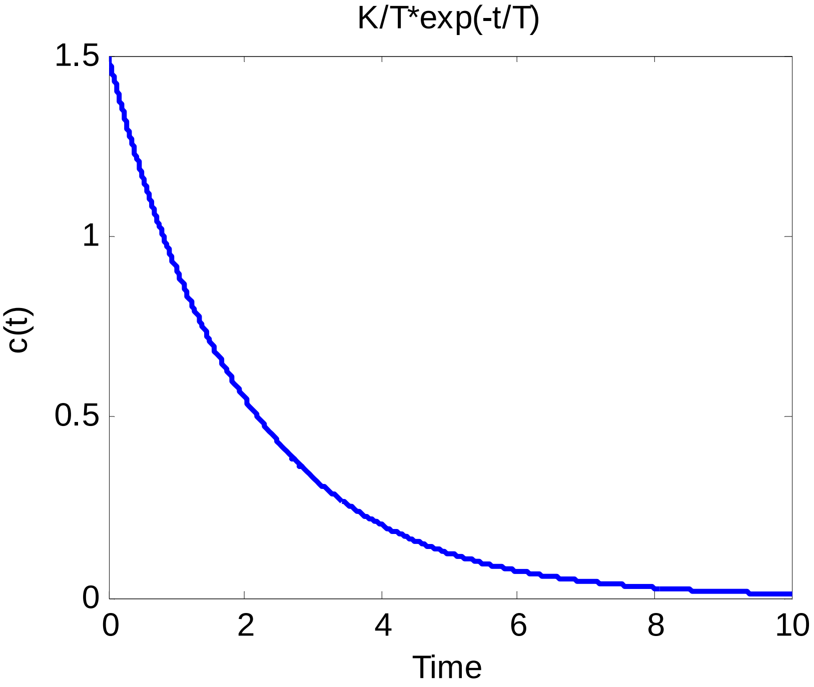
$$L^{-1}\left(\frac{C}{s+a}\right) = Ce^{-at}$$

$$c(t) = \frac{K}{T}e^{-t/T}$$

Impulse Response of 1st Order System

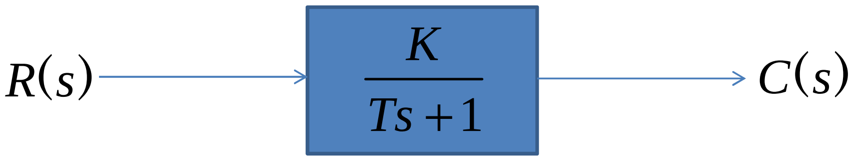
- If **K=3** and **T=2s** then

$$c(t) = \frac{K}{T}e^{-t/T}$$



Step Response of 1st Order System

- Consider the following 1st order system



$$R(s)\longrightarrow\boxed{\frac{K}{Ts+1}}\longrightarrow C(s)$$

$$R(s)=U(s)=\frac{1}{s}$$

$$C(s)=\frac{K}{s(Ts+1)}$$

- In order to find out the inverse Laplace of the above equation, we need to break it into partial fraction expansion

Forced Response **Natural Response**

$$C(s)=\frac{K}{s}-\frac{KT}{Ts+1}$$

Step Response of 1st Order System

$$C(s)=K\left(\frac{1}{s}-\frac{T}{Ts+1}\right)$$

- Taking Inverse Laplace of above equation

$$c(t)=K\left(u(t)-e^{-t/T}\right)$$

- Where **u(t)=1**

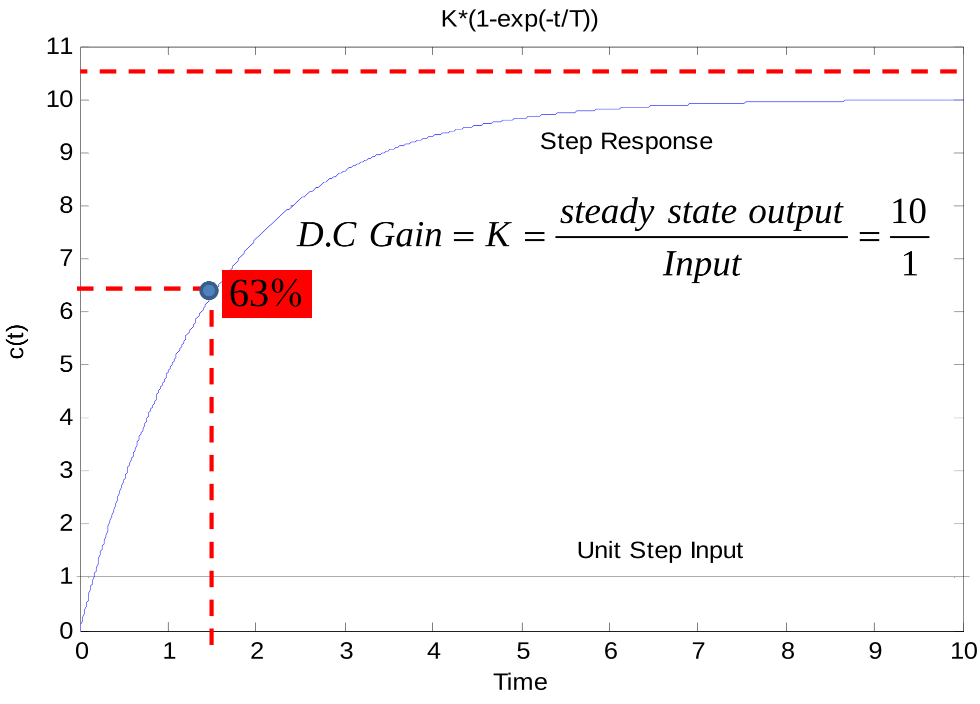
$$c(t)=K\left(1-e^{-t/T}\right)$$

- When **t=T**

$$c(t)=K\left(1-e^{-1}\right)=0.632K$$

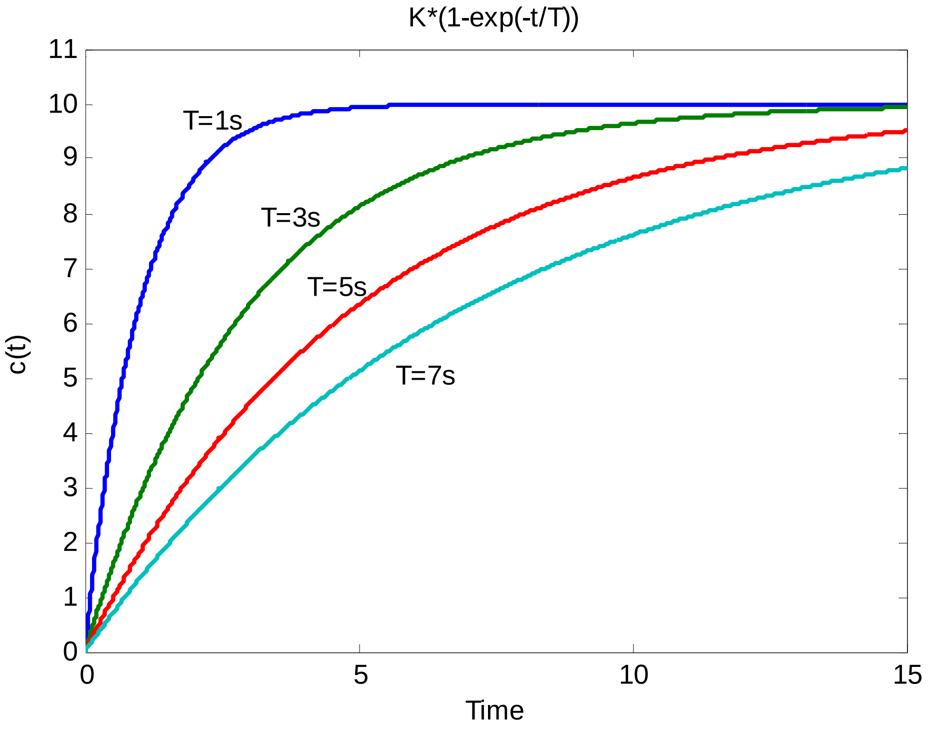
Step Response of 1st Order System

- If **K=10** and **T=1.5s** then $c(t) = K\left(1 - e^{-t/T}\right)$



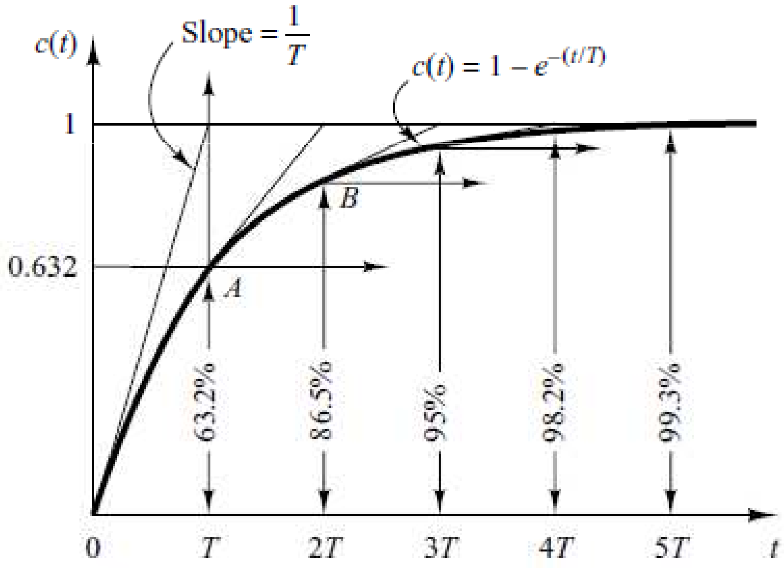
Step Response of 1st Order System

- If **K=10** and **T=1, 3, 5, 7** $c(t) = K\left(1 - e^{-t/T}\right)$



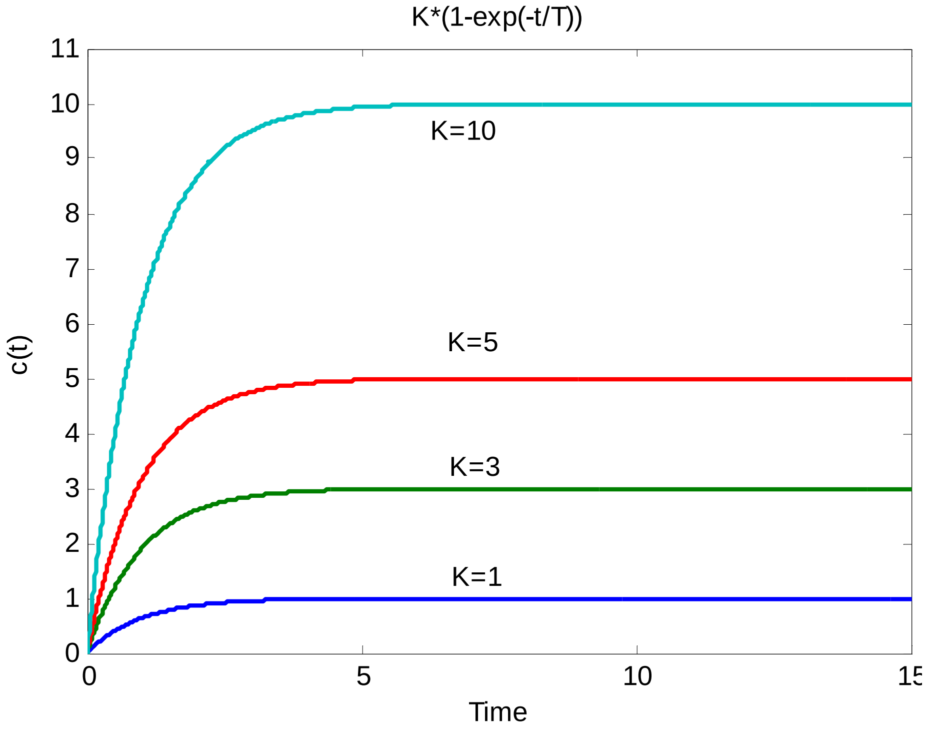
Step Response of 1st order System

- System takes five time constants to reach its final value.



Step Response of 1st Order System

- If **K=1, 3, 5, 10** and **T=1** $c(t) = K\left(1 - e^{-t/T}\right)$



Example -1

- Impulse response of a **1st** order system is given below.

$$c(t) = 3e^{-0.5t}$$

- **Find out**
  - Time constant T
  - D.C Gain K
  - Transfer Function
  - Step Response

Example -1

- The Laplace Transform of Impulse response of a system is actually the transfer function of the system.
- Therefore taking Laplace Transform of the impulse response given by following equation.

$$c(t) = 3e^{-0.5t}$$

$$C(s) = \frac{3}{S+0.5} \times 1 = \frac{3}{S+0.5} \times \delta(s)$$

$$\frac{C(s)}{\delta(s)} = \frac{C(s)}{R(s)} = \frac{3}{S+0.5}$$

$$\frac{C(s)}{R(s)} = \frac{6}{2S+1}$$

Example -1

- Impulse response of a 1st order system is given below.

$$c(t) = 3e^{-0.5t}$$

- Find out
  - Time constant **T=2**
  - D.C Gain **K=6**
  - Transfer Function
  - Step Response
  - Also Draw the Step response on your notebook

$$\frac{C(s)}{R(s)} = \frac{6}{2S + 1}$$

Example -1

- For step response integrate impulse response

$$c(t) = 3e^{-0.5t}$$

$$\int c(t)dt = 3\int e^{-0.5t}dt$$

$$c_s(t) = -6e^{-0.5t} + C$$

- We can find out **C** if initial condition is known e.g. **$c_s(0)=0$**

$$0 = -6e^{-0.5\times 0} + C$$
$$C = 6$$

$$c_s(t) = 6 - 6e^{-0.5t}$$

Example –1

- If initial Conditions are not known then partial fraction expansion is a better choice

$$\frac{C(s)}{R(s)}=\frac{6}{2S+1}$$

since $R(s)$ is a step input, $R(s)=\frac{1}{s}$

$$C(s)=\frac{6}{s(2S+1)}$$

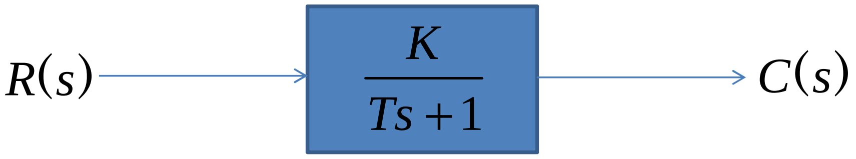
$$\frac{6}{s(2S+1)}=\frac{A}{s}+\frac{B}{2s+1}$$

$$\frac{6}{s(2S+1)}=\frac{6}{s}-\frac{6}{s+0.5}$$

$$c(t)=6-6e^{-0.5t}$$

Ramp Response of 1st Order System

- Consider the following 1st order system



$$R(s)=\frac{1}{s^2}$$

$$C(s)=\frac{K}{s^2(Ts+1)}$$

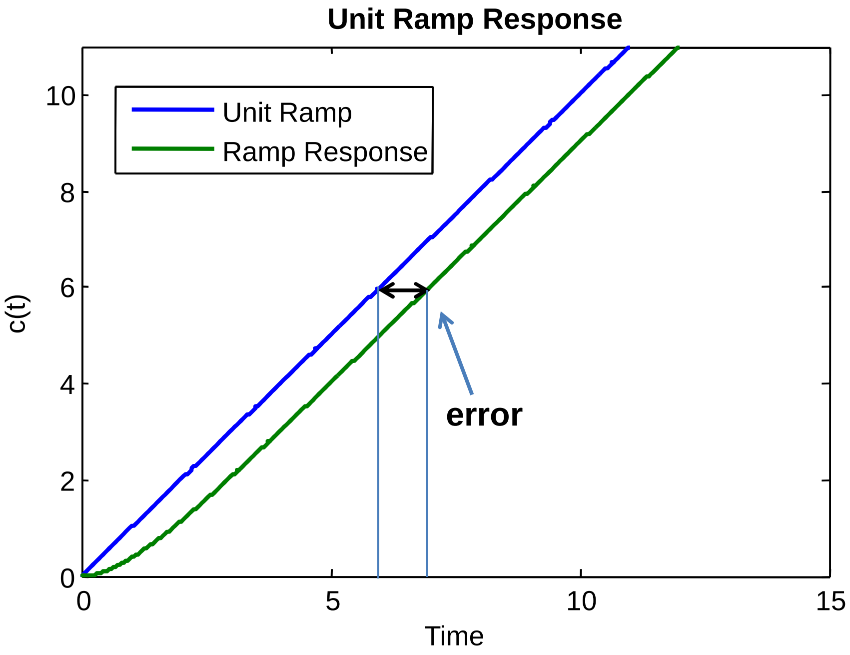
- The ramp response is given as

$$c(t)=K\left(t-T+Te^{-t/T}\right)$$

Ramp Response of 1st Order System

- If **K=1** and **T=1**

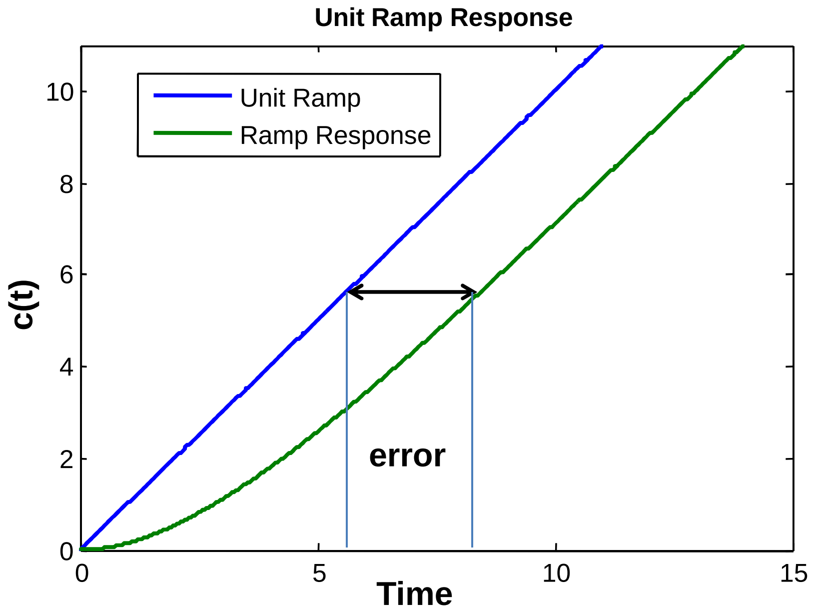
$$c(t) = K\left(t - T + Te^{-t/T}\right)$$



Ramp Response of 1st Order System

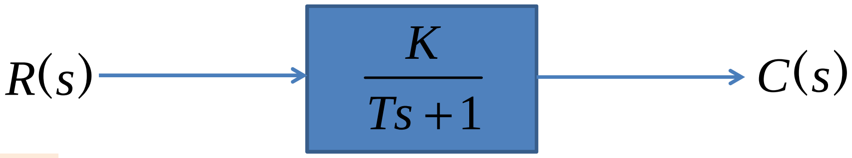
- If **K=1** and **T=3**

$$c(t) = K\left(t - T + Te^{-t/T}\right)$$



Parabolic Response of 1st Order System

- Consider the following 1st order system



Unit Parabolic

$$R(s) = \frac{2}{s^3}$$

Therefore,

$$C(s) = \frac{K}{s^3(Ts+1)}$$

- **Do it yourself**

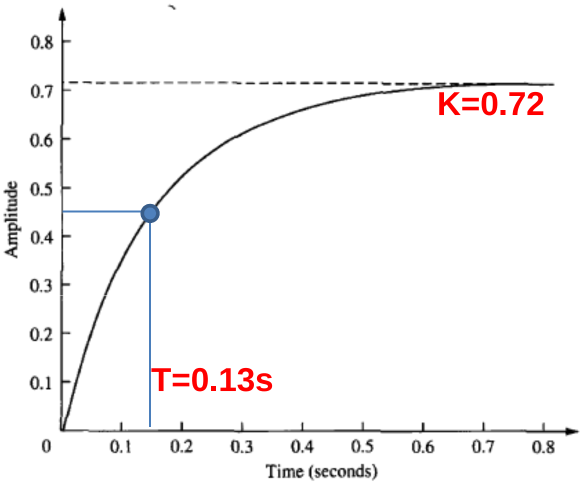
Practical Determination of Transfer Function of 1st Order Systems

- If we can identify *T* and *K* from laboratory testing we can obtain the transfer function of the system.

$$\frac{C(s)}{R(s)} = \frac{K}{Ts+1}$$

Practical Determination of Transfer Function of 1st Order Systems

- For example, assume the unit step response given in figure.
- From the response, we can measure the time constant, that is, the time for the amplitude to reach 63% of its final value.
- Since the final value is about 0.72 the time constant is evaluated where the curve reaches 0.63 x 0.72 = 0.45, or about **0.13** second.
- **K** is simply steady state value.



- Thus transfer function is obtained as:

$$\frac{C(s)}{R(s)}=\frac{0.72}{0.13s+1}=\frac{5.5}{s+7.7}$$

1st Order System with a Zero

$$\frac{C(s)}{R(s)}=\frac{K(1+\alpha s)}{Ts+1}$$

- Zero of the system lie at *-1/α* and pole at *-1/T*.
- Step response of the system would be:

$$C(s)=\frac{K(1+\alpha s)}{s(Ts+1)}$$

$$C(s)=\frac{K}{s}+\frac{K(\alpha-T)}{(Ts+1)}$$

$$c(t)=K\left(1-e^{-t/T}\right) \quad c(t)=K+\frac{K}{T}(\alpha-T)e^{-t/T}$$

1^{st} Order System with & W/O Zero

$$\frac{C(s)}{R(s)} = \frac{K}{Ts+1} \qquad \frac{C(s)}{R(s)} = \frac{K(1+\alpha s)}{Ts+1}$$

$$c(t) = K\left(1-e^{-t/T}\right) \qquad c(t) = K + \frac{K}{T}(\alpha - T)e^{-t/T}$$

- If ***T>α*** the response will be same

$$c(t) = K + \frac{K}{T}(-n)e^{-t/T}$$

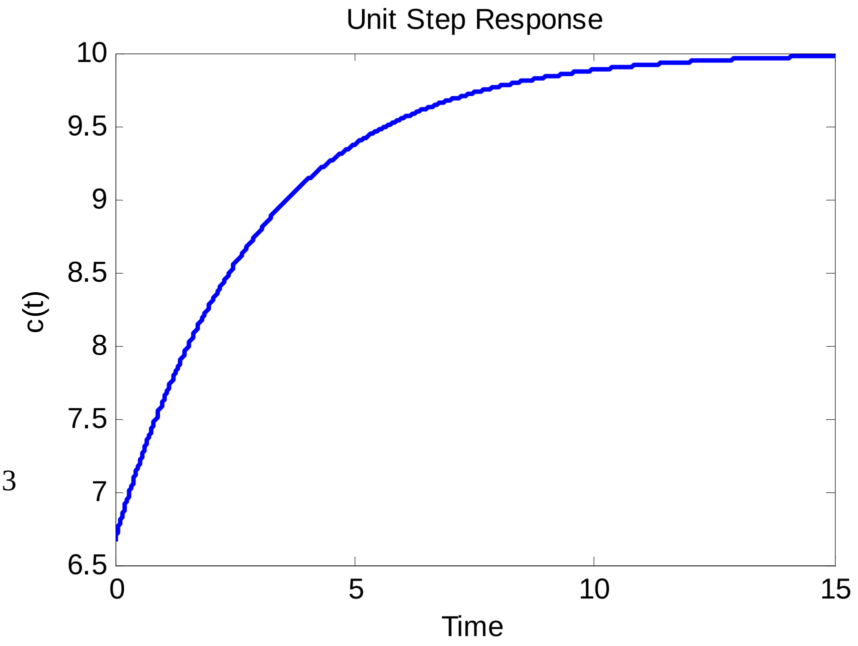
$$c(t) = K\left(1 - \frac{Kn}{T}e^{-t/T}\right)$$

1^{st} Order System with & W/O Zero

- If *T>α* the response of the system would look like

$$\frac{C(s)}{R(s)} = \frac{10(1+2s)}{3s+1}$$

$$c(t) = 10 + \frac{10}{3}(2-3)e^{-t/3}$$

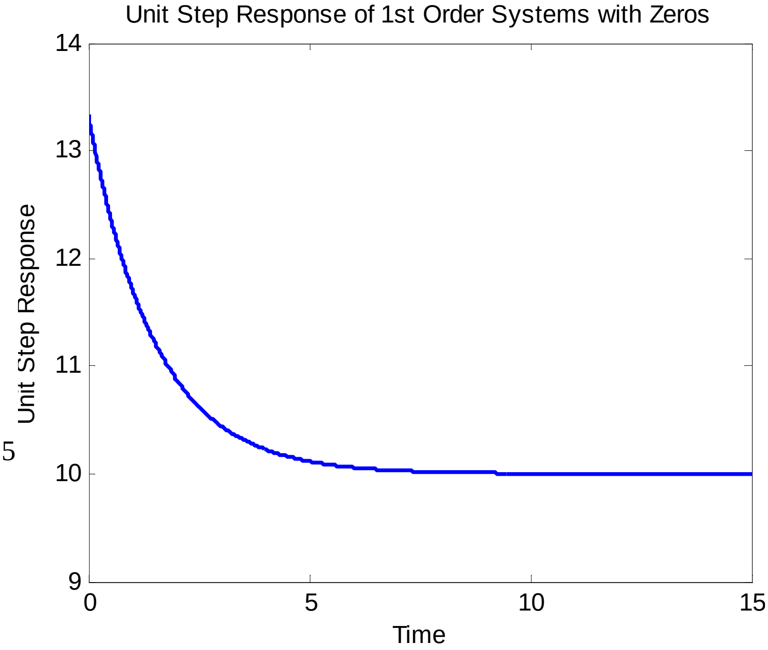


1 st Order System with & W/O Zero

- If *T<α* the response of the system would look like

$$\frac{C(s)}{R(s)} = \frac{10(1+2s)}{1.5s+1}$$

$$c(t) = 10 + \frac{10}{1.5}(2-1.5)e^{-t/1.5}$$



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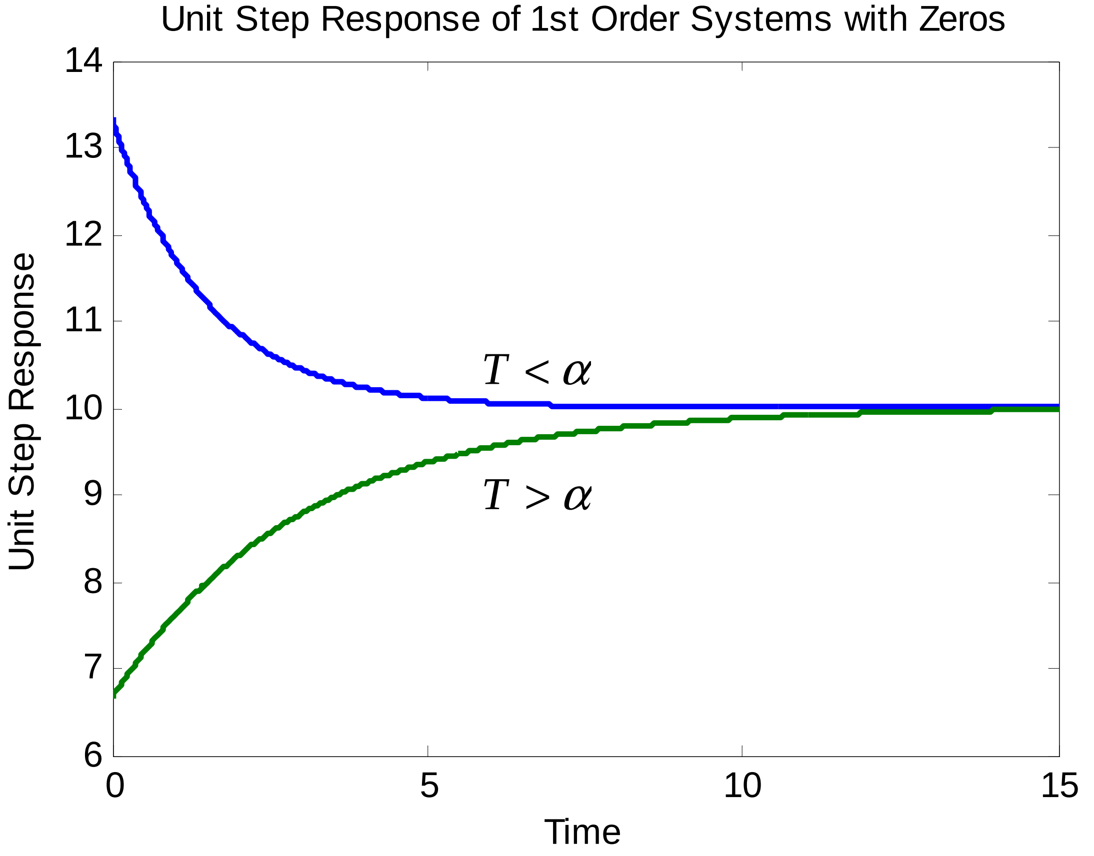
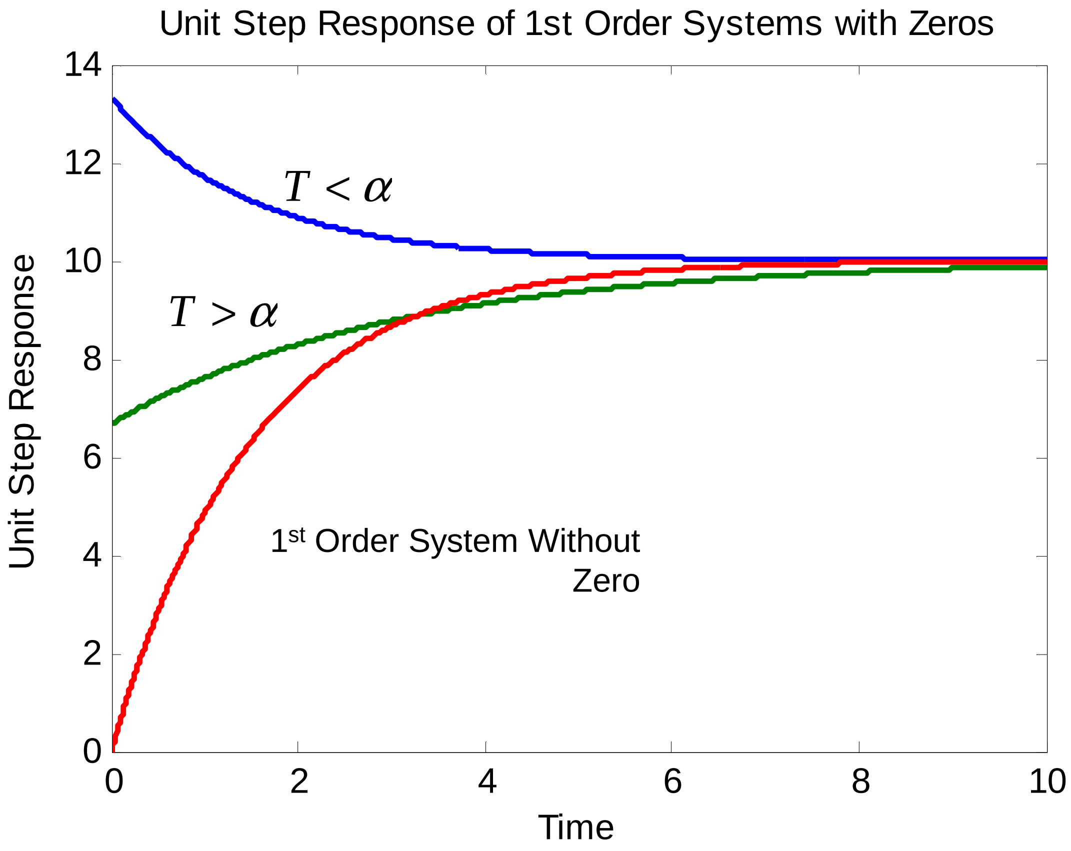
1 st Order System with a Zero

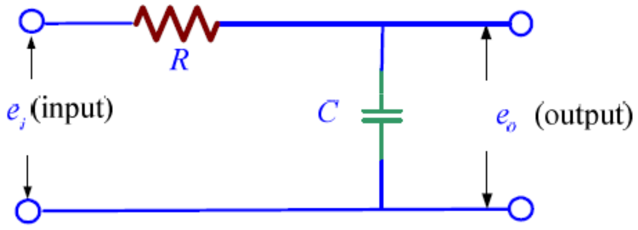


1^{st} Order System with & W/O Zero



Examples of First Order Systems

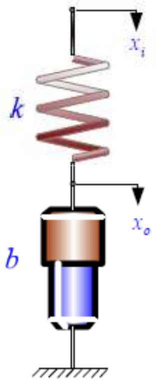
- Electrical System



$$\frac{E_o(s)}{E_i(s)} = \frac{1}{RCs+1}$$

Examples of First Order Systems

- Mechanical System



$$\frac{X_o(s)}{X_i(s)} = \frac{1}{\frac{b}{k}s + 1}$$