Automatic Control Theory

CSE 322

Lec. 2
Transfer Functions & Block Diagrams

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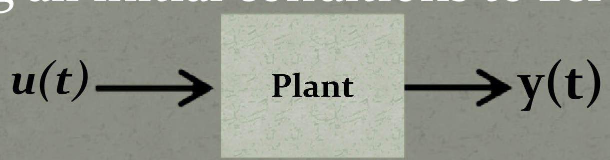
Transfer Function

- Transfer Function is the ratio of Laplace transform of the output to the Laplace transform of the input. Considering all initial conditions to zero.



$$If \quad \ell u(t) = U(S) \quad and$$

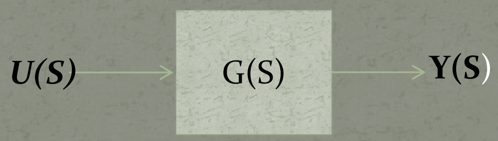
$$\ell y(t) = Y(S)$$

- Where ℓ is the Laplace operator.

Transfer Function

- Then the transfer function **G(S)** of the plant is given as

$$G(S) = \frac{Y(S)}{U(S)}$$



Why Laplace Transform?

- By use of Laplace transform we can convert many common functions into algebraic function of complex variable ***s***.
- For example

$$\ell \sin \omega t = \frac{\omega}{s^2 + \omega^2}$$

Or

$$\ell e^{-at} = \frac{1}{s+a}$$

- Where ***s*** is a complex variable (complex frequency) and is given as $s = \sigma + j\omega$

Laplace Transform of Derivatives

- Not only common function can be converted into simple algebraic expressions but calculus operations can also be converted into algebraic expressions.
- For example

$$\ell\frac{dx(t)}{dt} = sX(S) - x(0)$$

$$\ell\frac{d^2x(t)}{dt^2} = s^2X(S) - sx(0) - \frac{dx(0)}{dt}$$

Laplace Transform of Derivatives

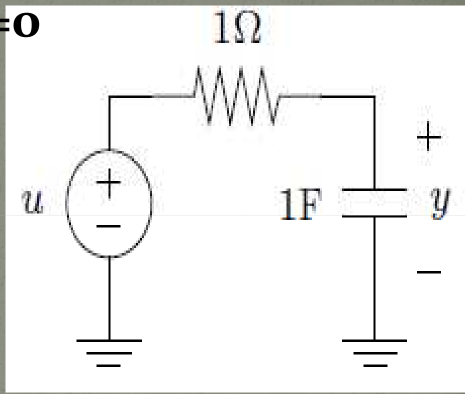
- In general

$$\ell\frac{d^nx(t)}{dt^n} = s^nX(S) - s^{n-1}x(0) - \cdots - x^{n-1}(0)$$

- Where $x(0)$ is the initial condition of the system.

Example: RC Circuit

- ***u*** is the input voltage applied at **t=0**
- **y** is the capacitor voltage



- If the capacitor is not already charged then **y(0)=0**.

Laplace Transform of Integrals

$$\ell \int x(t)\,dt = \frac{1}{s} X(S)$$

- The time domain integral becomes division by **s** in frequency domain.

Calculation of the Transfer Function

- Consider the following **ODE** where ***y(t)*** is input of the system and **x(t)** is the output.

$$A\frac{d^2x(t)}{dt^2} = C\frac{dy(t)}{dt} - B\frac{dx(t)}{dt}$$

- or

$$Ax''(t) = Cy'(t) - Bx'(t)$$

- Taking the Laplace transform on either sides

$$A[s^2X(s) - sx(0) - x'(0)] = C[sY(s) - y(0)] - B[sX(s) - x(0)]$$

Calculation of the Transfer Function

$$A[s^2X(s) - sx(0) - x'(0)] = C[sY(s) - y(0)] - B[sX(s) - x(0)]$$

- Considering Initial conditions to zero in order to find the transfer function of the system

$$As^2X(s) = CsY(s) - BsX(s)$$

- Rearranging the above equation

$$As^2X(s) + BsX(s) = CsY(s)$$

$$X(s)[As^2 + Bs] = CsY(s)$$

$$\frac{X(s)}{Y(s)} = \frac{Cs}{As^2 + Bs} = \frac{C}{As + B}$$

Example

1. Find out the transfer function of the RC network shown in figure-1. Assume that the capacitor is not initially charged.

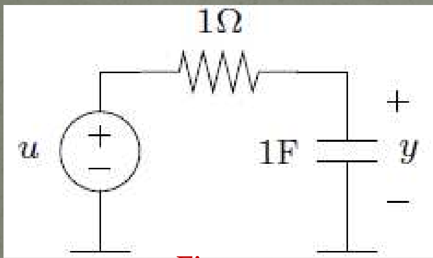


Figure-1

$$y'(t) + y(t) = u(t)$$

2. ***u(t)*** and ***y(t)*** are the input and output respectively of a system defined by following ODE. **Determine** the Transfer Function. Assume there is no any energy stored in the system.

$$6u''(t) - 3u(t) + \int y(t)dt = -3y'''(t) - y(t)$$

Transfer Function

- In general

$$a_0 \overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(m)}{x} + b_1 \overset{(m-1)}{x} + \cdots + b_{m-1}\dot{x} + b_m x \qquad (n \geq m)$$

- Where ***x*** is the input of the system and **y** is the output of the system.

$$\text{Transfer function} = G(s) = \left.\frac{\mathscr{L}[\text{output}]}{\mathscr{L}[\text{input}]}\right|_{\text{zero initial conditions}}$$

$$= \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

Transfer Function

$$\frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} \qquad (n \geq m)$$

- When order of the denominator polynomial is greater than the numerator polynomial the transfer function is said to be '**proper**'.
- Otherwise '**improper**'

Transfer Function

- Transfer function helps us to check
  - The stability of the system
  - Time domain and frequency domain characteristics of the system
  - Response of the system for any given input

Stability of Control System

- There are several meanings of stability, in general there are two kinds of stability definitions in control system study.
  - Absolute Stability
  - Relative Stability

Stability of Control System

$$\frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

- Roots of denominator polynomial of a transfer function are called '**poles**'.
- And the roots of numerator polynomials of a transfer function are called '**zeros**'.

Stability of Control System

- Poles of the system are represented by '**x**' and zeros of the system are represented by '**o**'.
- **System order** is always equal to number of poles of the transfer function.

- Following transfer function represents **n^th^** order plant.

$$\frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

Stability of Control System

- Poles is also defined as "it is the frequency at which system becomes infinite". Hence the name pole where field is infinite.

$$\frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

- And zero is the frequency at which system becomes **o**.

Example

- Consider the Transfer function calculated is.

$$G(s) = \frac{X(s)}{Y(s)} = \frac{C}{As + B}$$

the denominator polynomial is $As + B = 0$

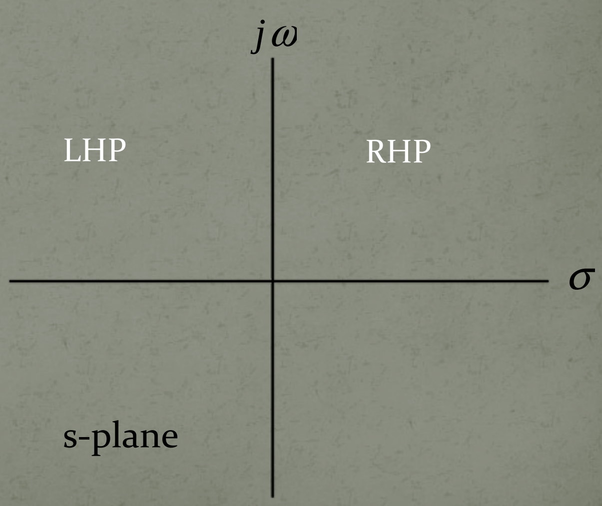
- The only pole of the system is

$$s = -\frac{B}{A}$$

Stability of Control Systems

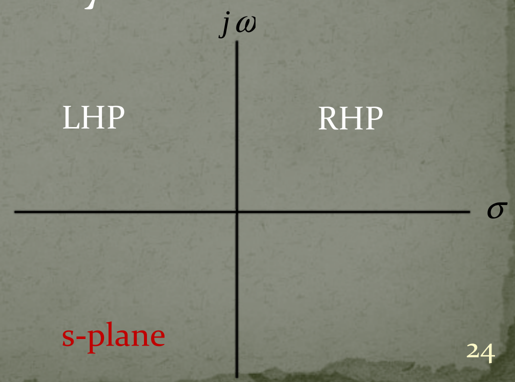
- The poles and zeros of the system are plotted in s-plane to check the stability of the system.

Recall $s = \sigma + j\omega$



Stability of Control Systems

- If all the poles of the system lie in left half plane the system is said to be **Stable**.
- If any of the poles lie in right half plane the system is said to be **Unstable**.
- If pole(s) lie on imaginary axis the system is said to be **Marginally Stable**.



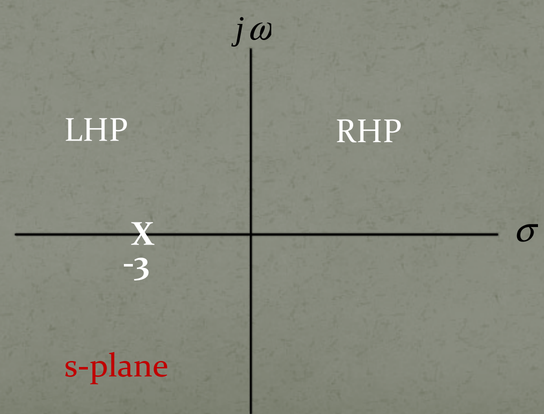
Stability of Control Systems

- For example

$$G(s) = \frac{C}{As + B}, \quad \text{if } A = 1, B = 3 \textit{ and } C = 10$$

- Then the only pole of the system lie at

$$pole = -3$$



Examples

- Consider the following transfer functions.
  - Determine whether the transfer function is **proper** or **improper**
  - Calculate the Poles and zeros of the system
  - Determine the order of the system
  - Draw the pole-zero map
  - Determine the Stability of the system

i) $$G(s) = \frac{s+3}{s(s+2)}$$

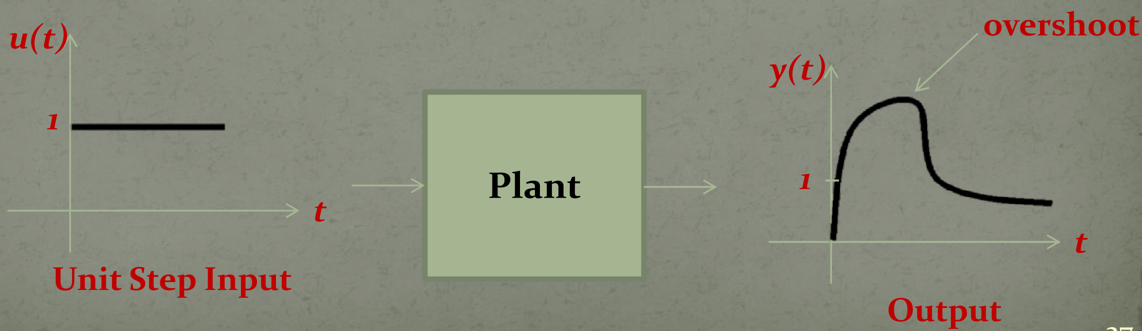
ii) $$G(s) = \frac{s}{(s+1)(s+2)(s+3)}$$

iii) $$G(s) = \frac{(s+3)^2}{s(s^2+10)}$$

iv) $$G(s) = \frac{s^2(s+1)}{s(s+10)}$$

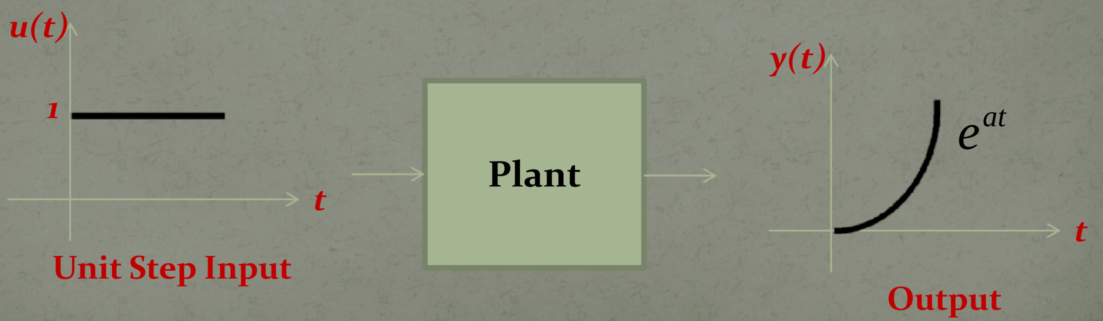
Another definition of Stability

- The system is said to be stable if for any bounded input the output of the system is also bounded (BIBO).
- Thus the for any bounded input the output either remain constant or decrease with time.



Another definition of Stability

- If for any bounded input the output is not bounded the system is said to be unstable.

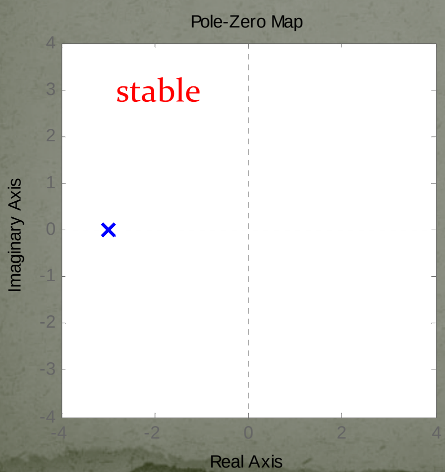


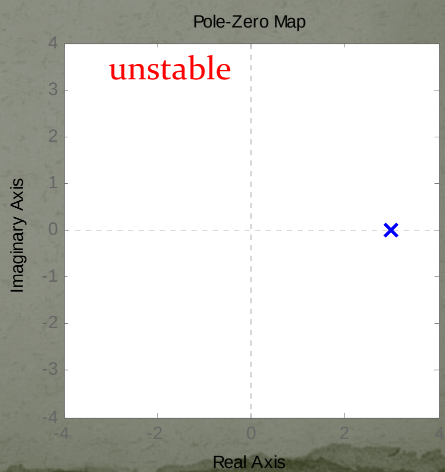
BIBO vs Transfer Function

- For example

$$G_1(s)=\frac{Y(s)}{U(s)}=\frac{1}{s+3}$$

$$G_2(s)=\frac{Y(s)}{U(s)}=\frac{1}{s-3}$$





BIBO vs Transfer Function

- For example

$$G_1(s) = \frac{Y(s)}{U(s)} = \frac{1}{s+3}$$

$$G_2(s) = \frac{Y(s)}{U(s)} = \frac{1}{s-3}$$

$$\ell^{-1}G_1(s) = \ell^{-1}\frac{Y(s)}{U(s)} = \ell^{-1}\frac{1}{s+3}$$

$$= y(t) = e^{-3t}u(t)$$

$$\ell^{-1}G_2(s) = \ell^{-1}\frac{Y(s)}{U(s)} = \ell^{-1}\frac{1}{s-3}$$

$$= y(t) = e^{3t}u(t)$$

BIBO vs Transfer Function

- For example

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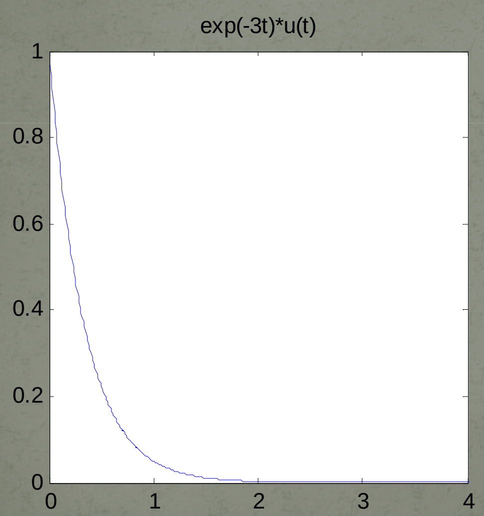
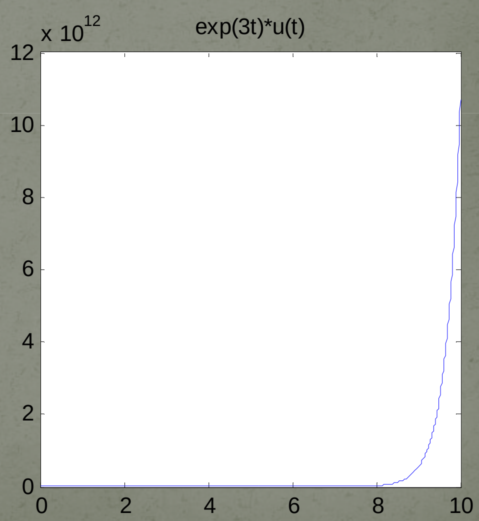


$$y(t) = e^{3t}u(t)$$



BIBO vs Transfer Function

- Whenever one or more than one poles are in RHP the solution of dynamic equations contains increasing exponential terms.
- Such as e^{3t}
- That makes the response of the system unbounded and hence the overall response of the system is unstable.

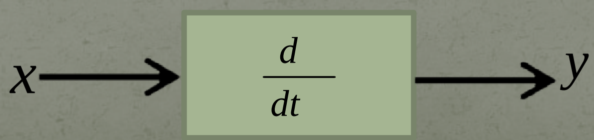
Block Diagram

Introduction

- A Block Diagram is a shorthand pictorial representation of the cause-and-effect **(i/p & o/p)** relationship of a system.

- The interior of the **rectangle** representing the block usually contains a description of or the name of the element, or the symbol for the mathematical operation to be performed on the input to yield the output.

- The arrows represent the direction of information or signal flow.



Summing Point

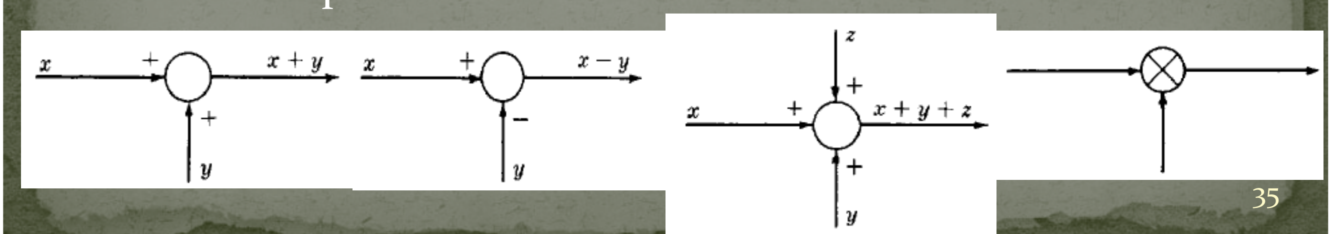
- The operations of addition and subtraction have a special representation.

- The block becomes a small circle, called a **summing point**, with the appropriate plus or minus sign associated with the arrows entering the circle.

- The output is the algebraic sum of the inputs.

- Any number of inputs may enter a summing point.

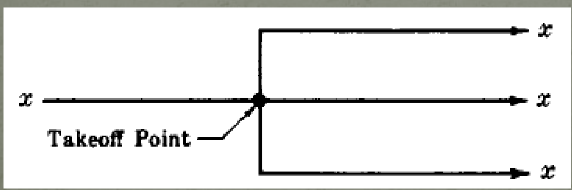
- Some books put a cross in the circle.

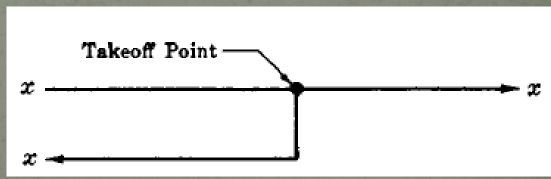


Takeoff Point (Node)

- In order to have the same signal or variable be an input to more than one block or summing point, a **Takeoff Point (Node)** is used.

- This permits the signal to proceed unaltered along several different paths to several destinations.

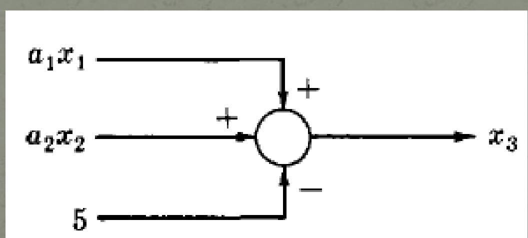


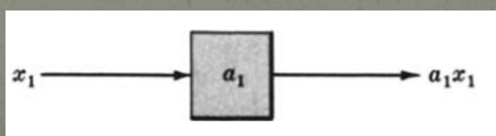


Example :-

- Consider the following equations in which $\mathbf{x_1}$, $\mathbf{x_2}$, $\mathbf{x_3}$, are variables, and $\boldsymbol{a_1}$, $\boldsymbol{a_2}$ *are* general coefficients or mathematical operators called **Gains**.

$$x_3 = a_1 x_1 + a_2 x_2 - 5$$





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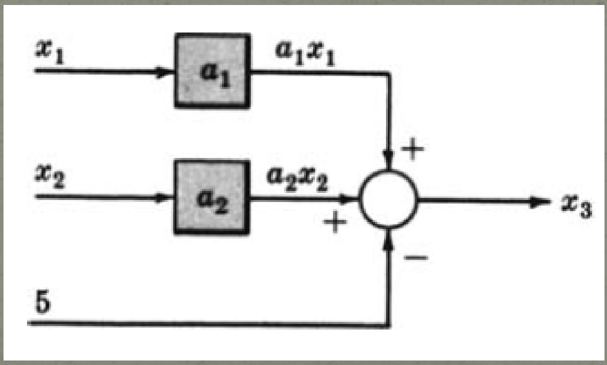


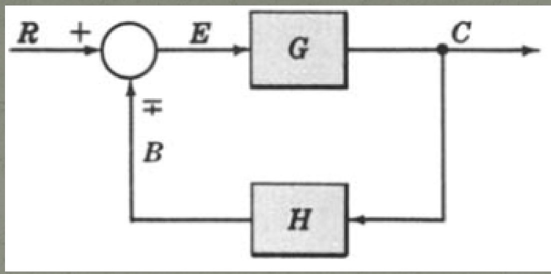
Example -2

- Draw the Block Diagrams of the following equations.

(1) $$x_2 = a_1\frac{dx_1}{dt} + \frac{1}{b}\int x_1 dt$$

(2) $$x_3 = a_1\frac{d^2x_2}{dt^2} + 3\frac{dx_1}{dt} - bx_1$$

Canonical Form of A Feedback Control System



$G \equiv$ direct transfer function $\equiv$ forward transfer function

$H \equiv$ feedback transfer function

$GH \equiv$ loop transfer function $\equiv$ open-loop transfer function

$C/R \equiv$ closed-loop transfer function $\equiv$ control ratio $\quad \frac{C}{R} = \frac{G}{1 \pm GH}$

$E/R \equiv$ actuating signal ratio $\equiv$ error ratio $\quad \frac{E}{R} = \frac{1}{1 \pm GH}$

[redacted]

Characteristic Equation

- The control ratio is the closed loop transfer function of the system.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}$$

- The denominator of closed loop transfer function determines the **characteristic equation** of the system.

- Which is usually determined as:

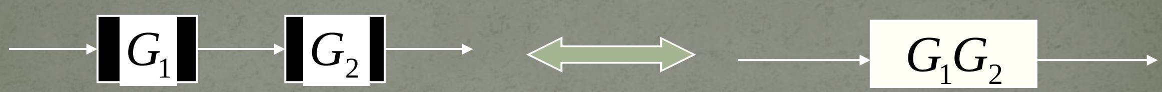
$$1 \pm G(s)H(s) = 0$$

Reduction of Complicated Block Diagrams

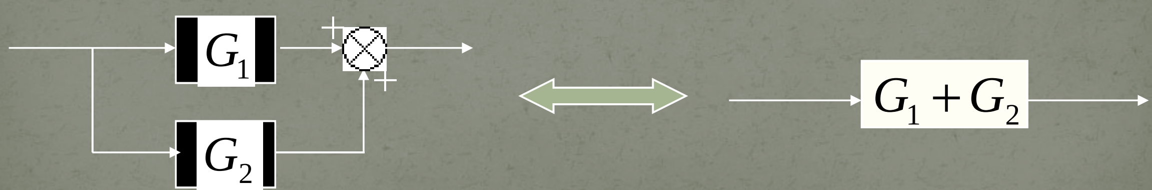
- The block diagram of a practical control system is often quite complicated.
- It may include several feedback or feedforward loops, and multiple inputs.
- By means of systematic block diagram reduction, every multiple loop linear feedback system may be reduced to canonical form.

Reduction Techniques

1. Combining blocks in cascade

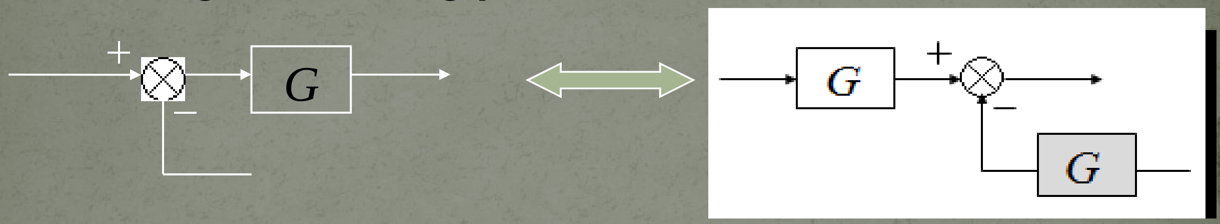


2. Combining blocks in parallel (Feed Forward)

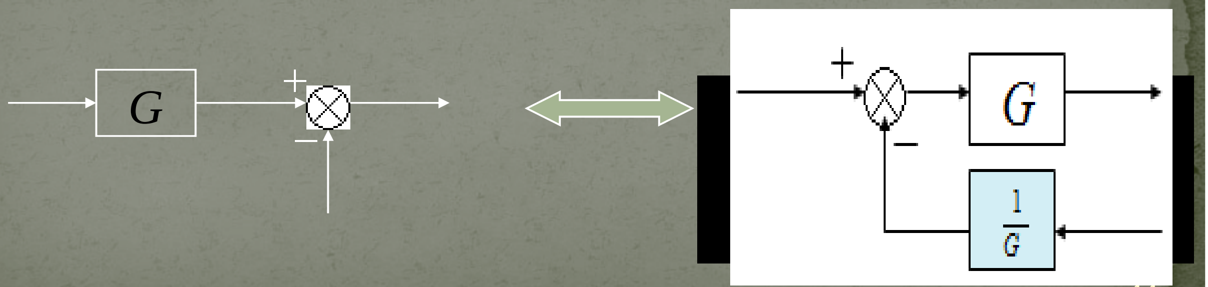


Reduction Techniques

3. Moving a summing point behind a block

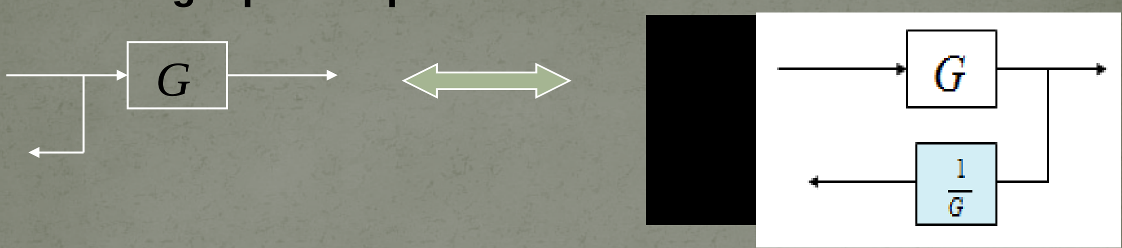


4. Moving a summing point ahead of a block

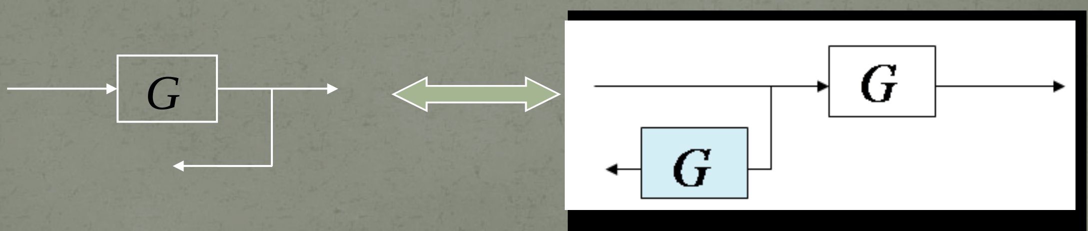


Reduction Techniques

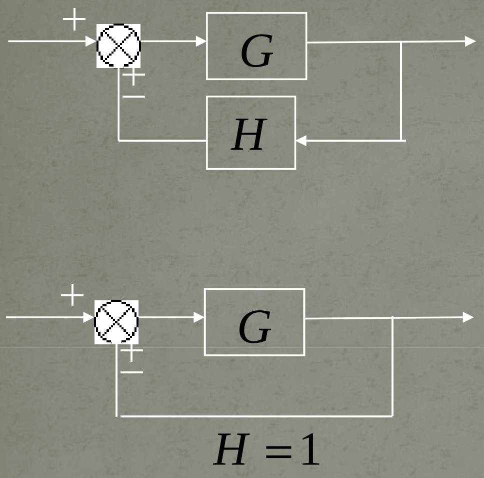
5. Moving a pickoff point behind a block

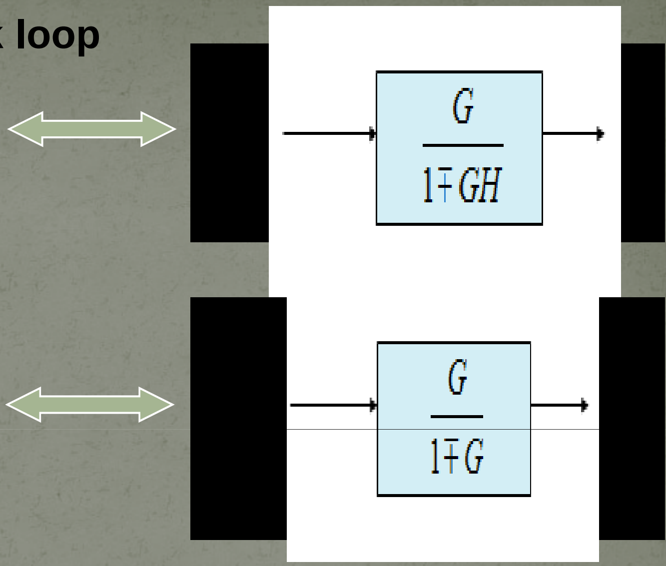


6. Moving a pickoff point ahead of a block

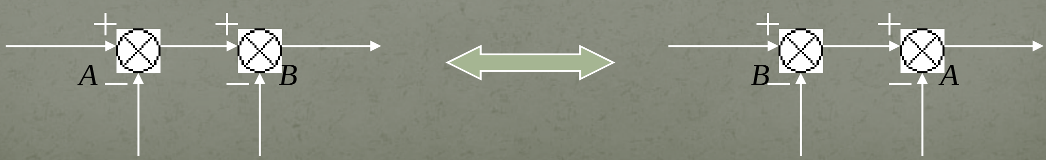


7. Eliminating a feedback loop

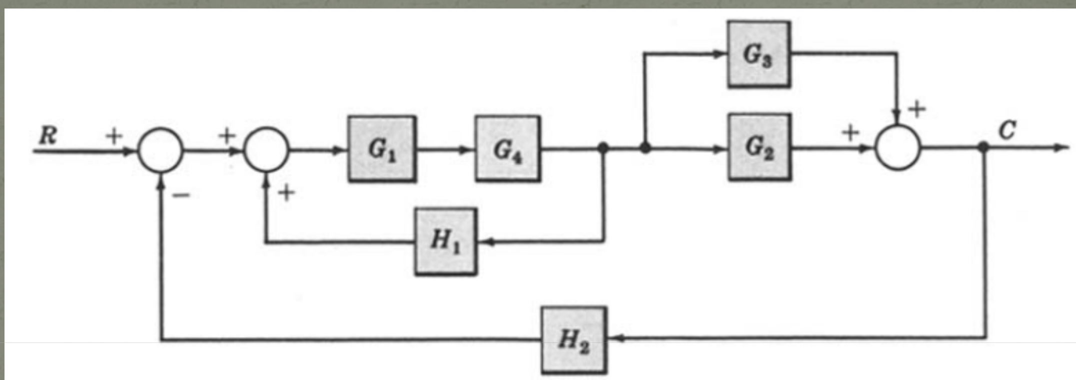




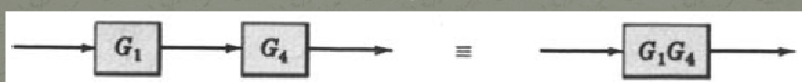
8. Swap with two neighboring summing points



Example -: Reduce the Following Block Diagram

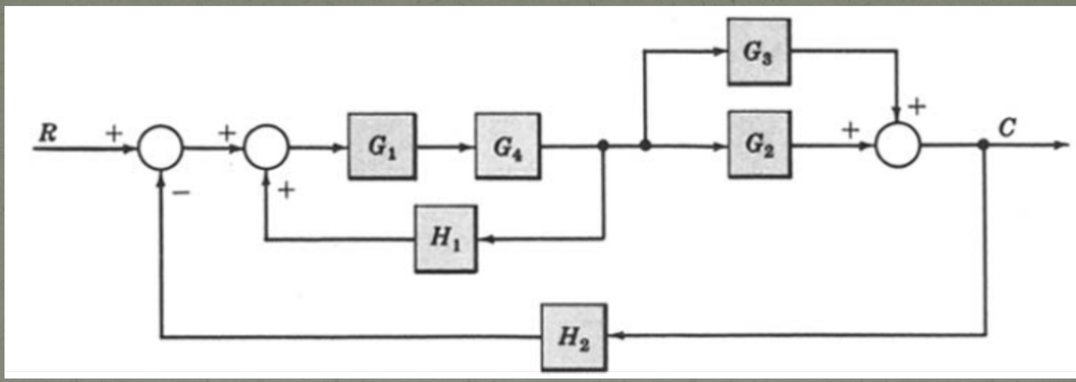


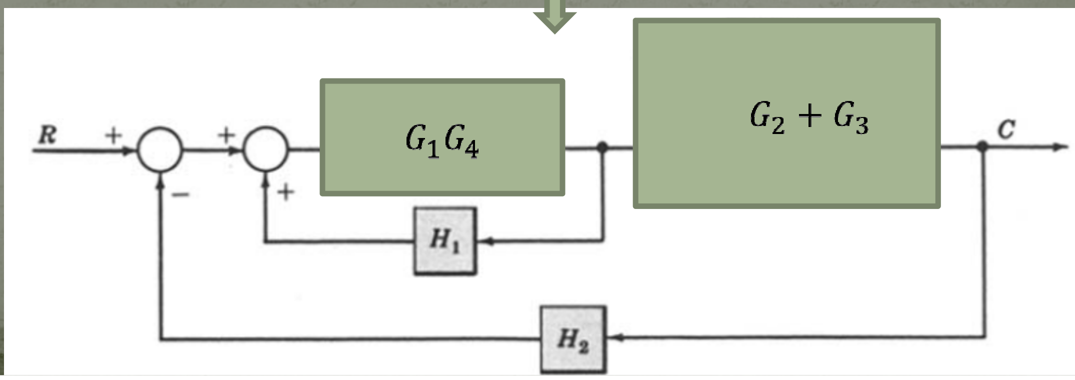
- Combine all cascade block using **rule-1**



- Combine all parallel block using **rule-2**

Example : Continued

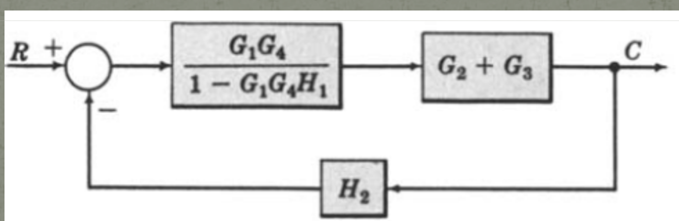




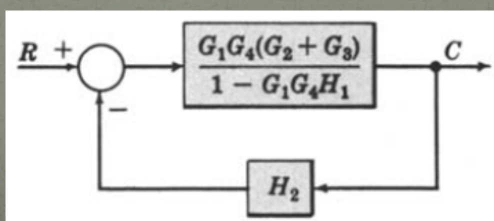
Example -: Continued

- Eliminate all minor feedback loops using **rule-7**

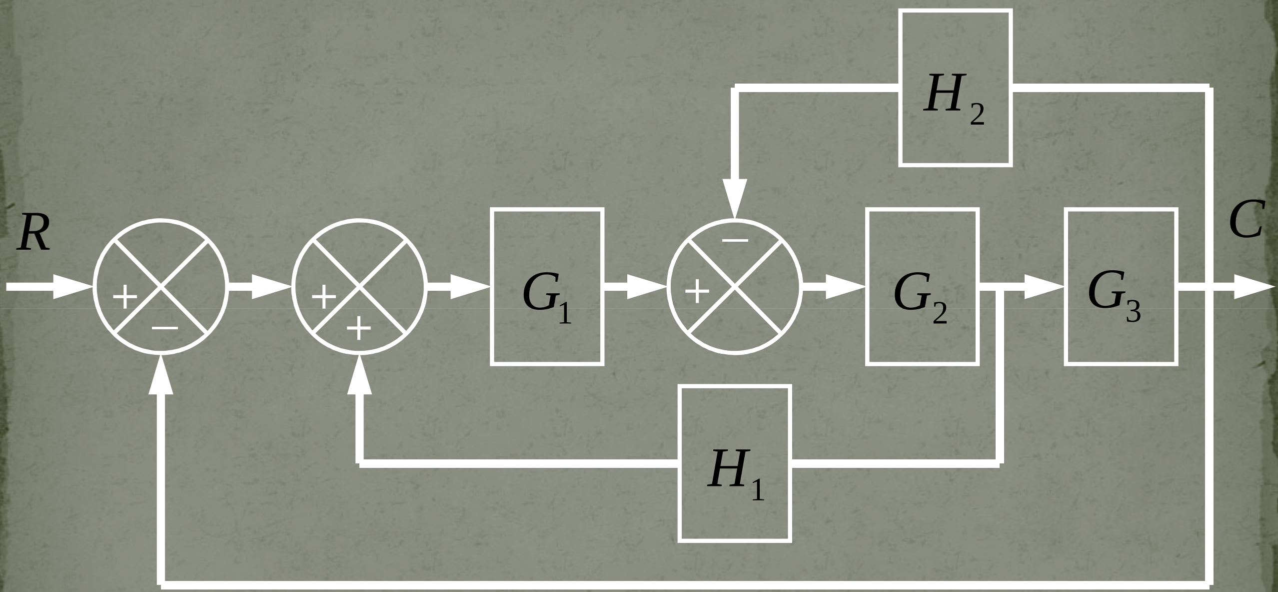
- After the elimination of minor feedback loop the block diagram is reduced to



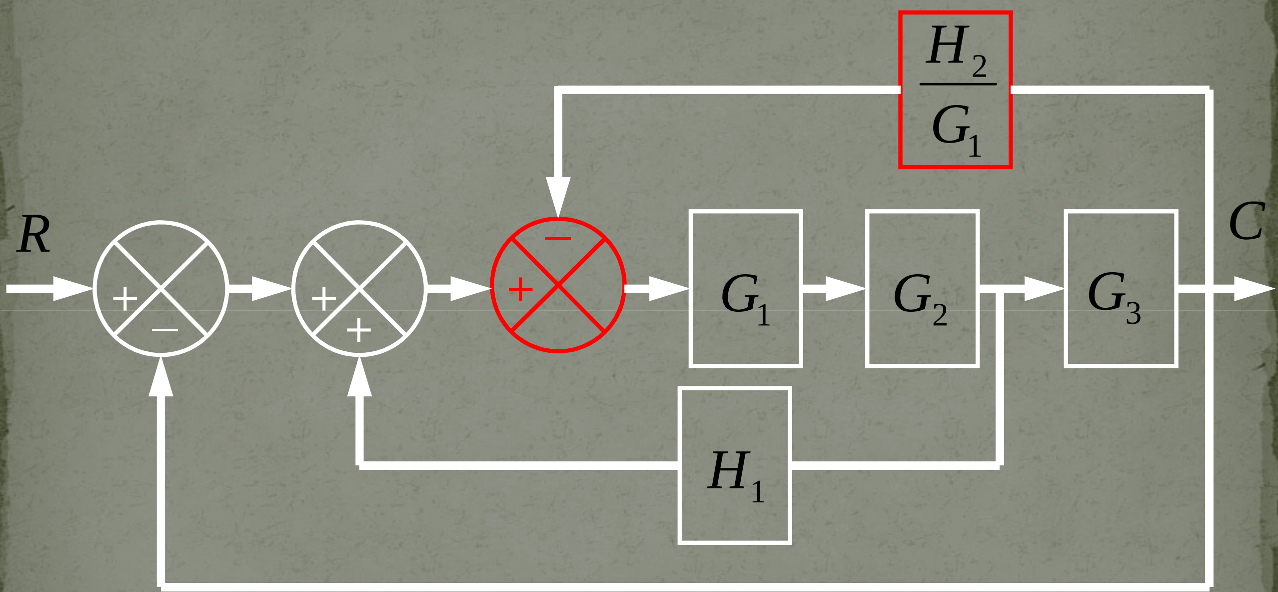
- Again blocks are in cascade are removed using **rule-1**



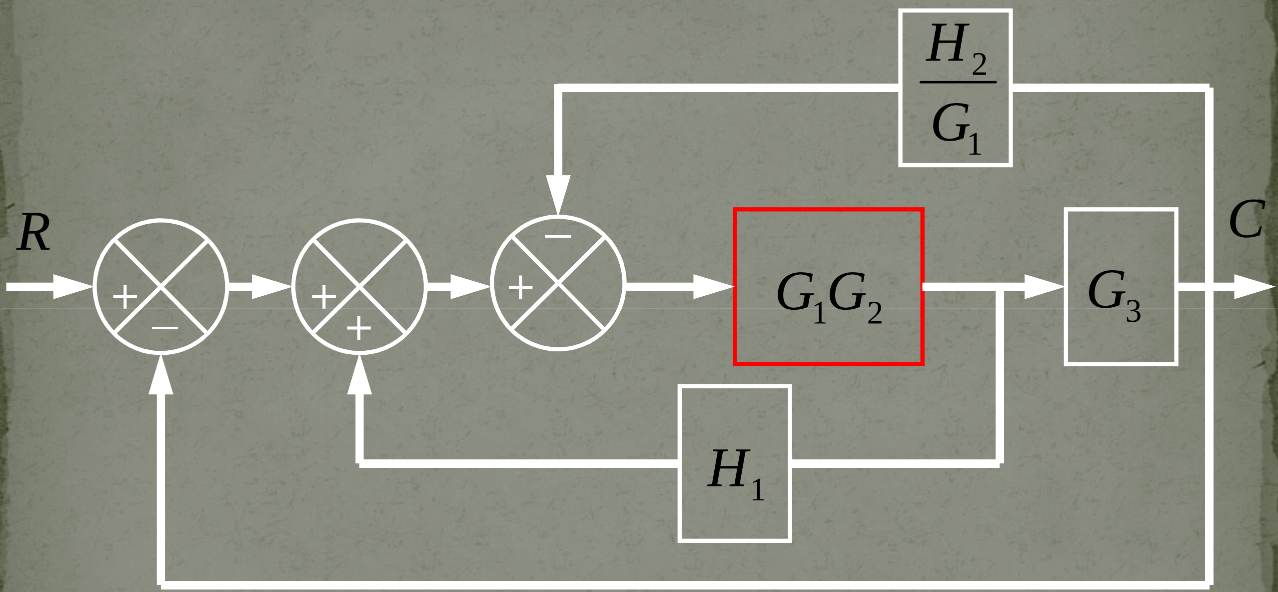
Example -4: Continued



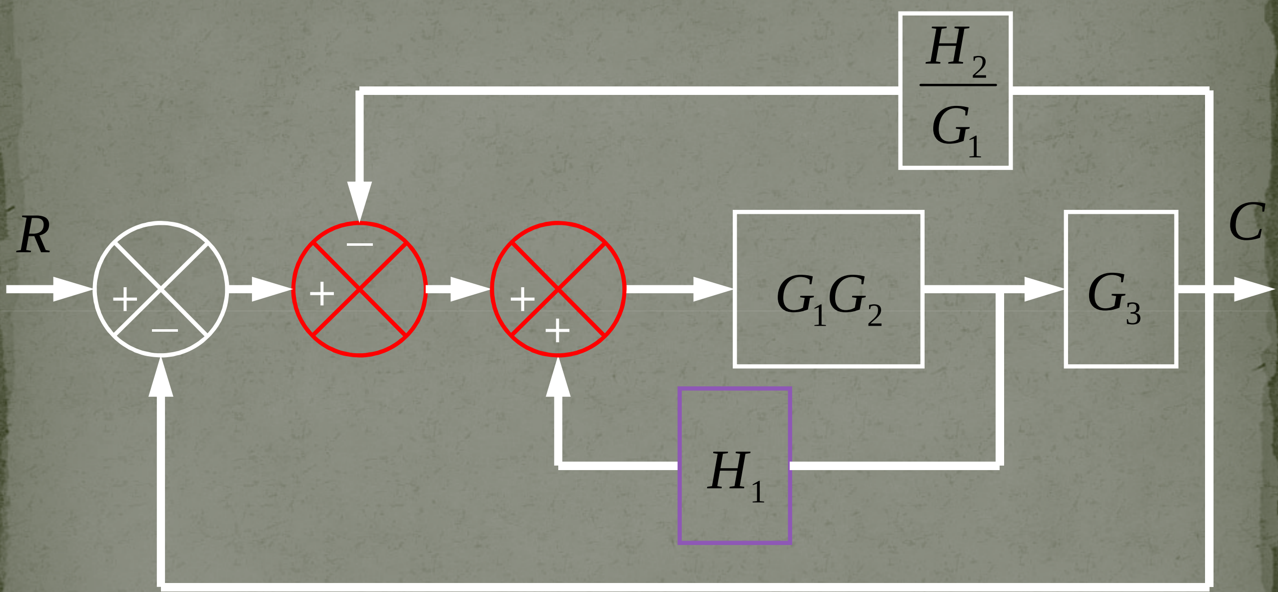
Example -4: Continued



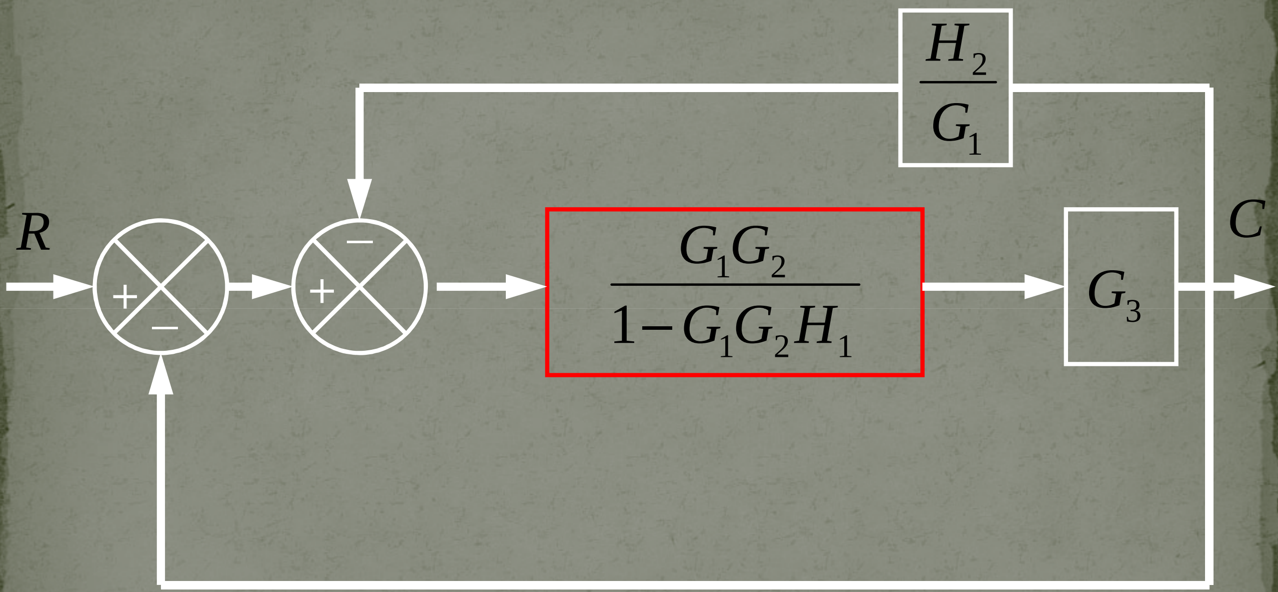
Example -4: Continued



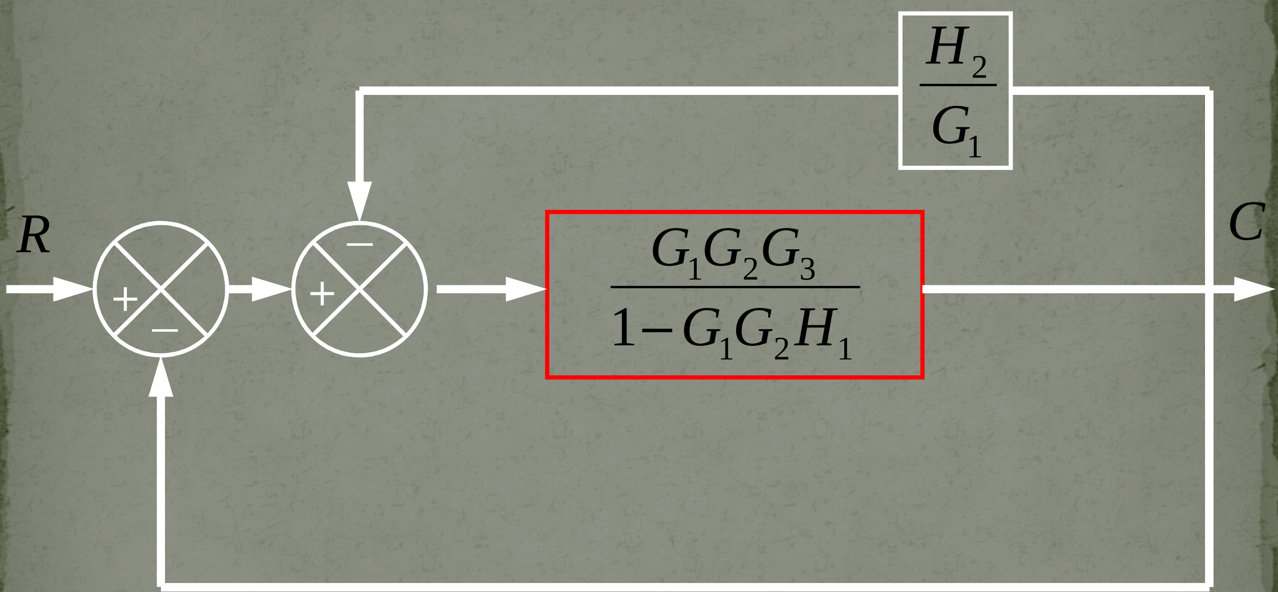
Example -4: Continued



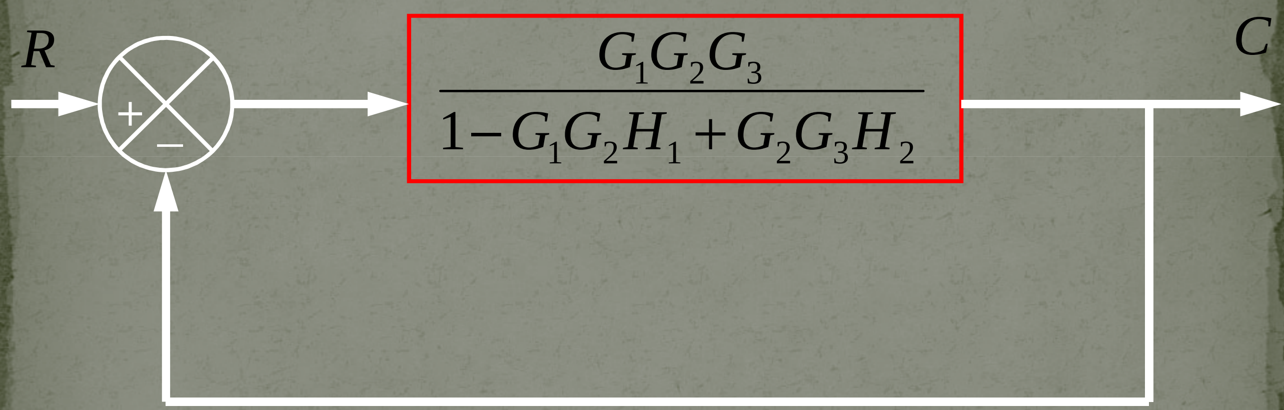
Example -4: Continued



Example -4: Continued

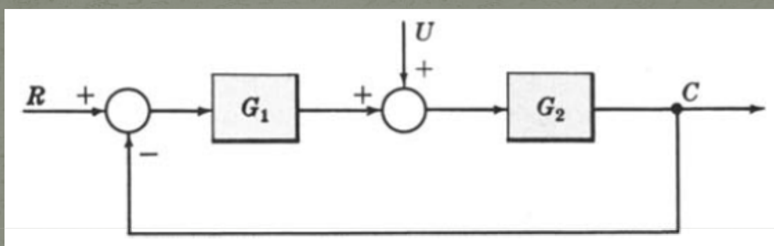


Example -4: Continued

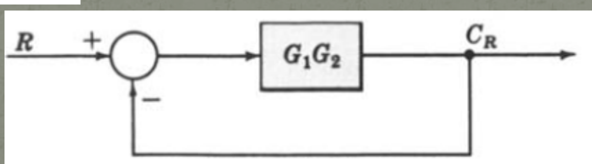


Example-5: Multiple Input System.

Determine the output C due to inputs R and U using the Superposition Method.

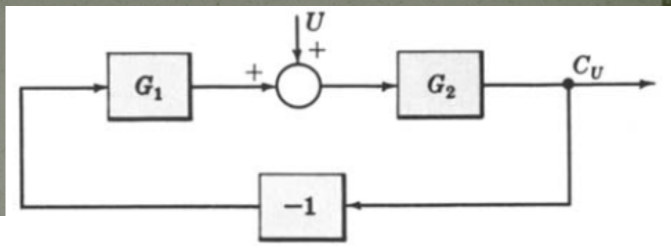


Step 1: Put $U \equiv 0$.
Step 2: The system reduces to



Step 3: the output C_R due to input R is $C_R = [G_1G_2/(1+G_1G_2)]R$.

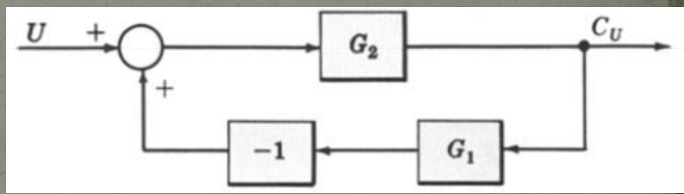
Example-5: Continued



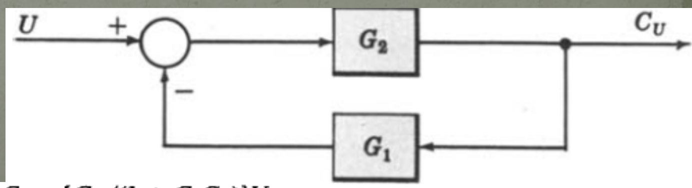
Step 4a: Put $R = 0$.

Step 4b: Put -1 into a block, representing the negative feedback effect:

Rearrange the block diagram:



Let the -1 block be absorbed into the summing point:



Step 4c: the output C_U due to input U is $C_U = [G_2/(1 + G_1G_2)]U$.

Example-5: Continued

Step 5: The total output is $C = C_R + C_U$

$$= \left[\frac{G_1G_2}{1 + G_1G_2}\right]R + \left[\frac{G_2}{1 + G_1G_2}\right]U$$

$$= \left[\frac{G_2}{1 + G_1G_2}\right][G_1R + U]$$