Automatic Control Theory

CSE 322

Lec. 11

Root Locus

Lecture Outline

- Construction of root loci
  - Angle and Magnitude Conditions
  - Illustrative Examples
- Closed loop stability via root locus
- Example of Root Locus

Construction of Root Loci

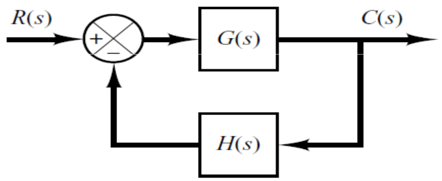
- Finding the roots of the characteristic equation of degree higher than **3** is laborious and will need computer solution.
- A simple method for finding the roots of the characteristic equation has been developed by **W. R. Evans** and used extensively in control engineering.
- This method, called the ***root-locus method***, is one in which the roots of the characteristic equation are plotted for all values of a system parameter.

Construction of Root Loci

- The roots corresponding to a particular value of this parameter can then be located on the resulting graph.
- Note that the parameter is usually **the gain**, but any **other variable of the open-loop transfer function** may be used.
- By using the root-locus method the designer can predict the effects on the location of the closed-loop poles of varying the gain value or adding open-loop poles and/or open-loop zeros.

Angle & Magnitude Conditions

- In constructing the root loci **angle** and **magnitude** conditions are important.
- Consider the system shown in following figure.



- The closed loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

Construction of Root Loci

- The characteristic equation is obtained by setting the denominator polynomial equal to zero.

$$1+G(s)H(s)=0$$

- Or

$$G(s)H(s)=-1$$

- Where ***G(s)H(s)*** is a ratio of polynomial in **s**.
- Since ***G(s)H(s)*** is a complex quantity it can be split into **angle** and **magnitude** part.

Angle & Magnitude Conditions

- The angle of ***G(s)H(s)=-1*** is

$$\angle G(s)H(s) = \angle -1$$
$$\angle G(s)H(s) = \pm 180^\circ(2k+1)$$

- Where ***k*=1,2,3…**
- The magnitude of ***G(s)H(s)=-1*** is

$$|G(s)H(s)| = |-1|$$
$$|G(s)H(s)| = 1$$

Angle & Magnitude Conditions

- Angle Condition

$$\angle G(s)H(s) = \pm 180^\circ(2k+1) \qquad (k=1,2,3...)$$

- Magnitude Condition

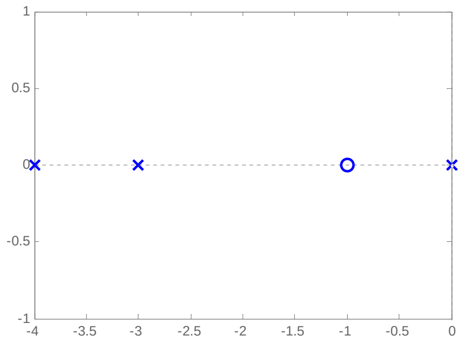
$$|G(s)H(s)| = 1$$

- The values of **s** that fulfill both the angle and magnitude conditions are the roots of the characteristic equation, or the closed-loop poles.
- A locus of the points in the complex plane satisfying the angle condition alone is the root locus.

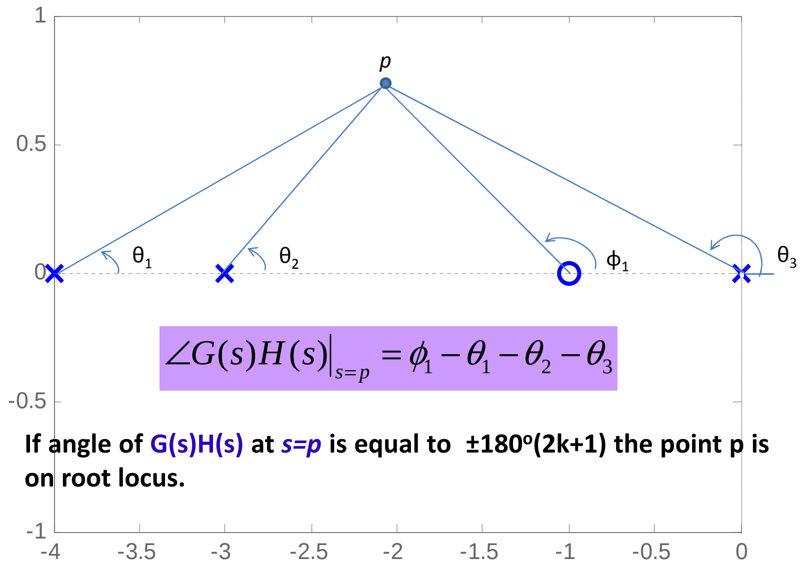
Angle and Magnitude Conditions (Graphically)

- To apply Angle and magnitude conditions graphically we must first draw the **poles** and **zeros** of **G(s)H(s)** in s-plane.
- For example if **G(s)H(s)** is given by

$$G(s)H(s) = \frac{s+1}{s(s+3)(s+4)}$$



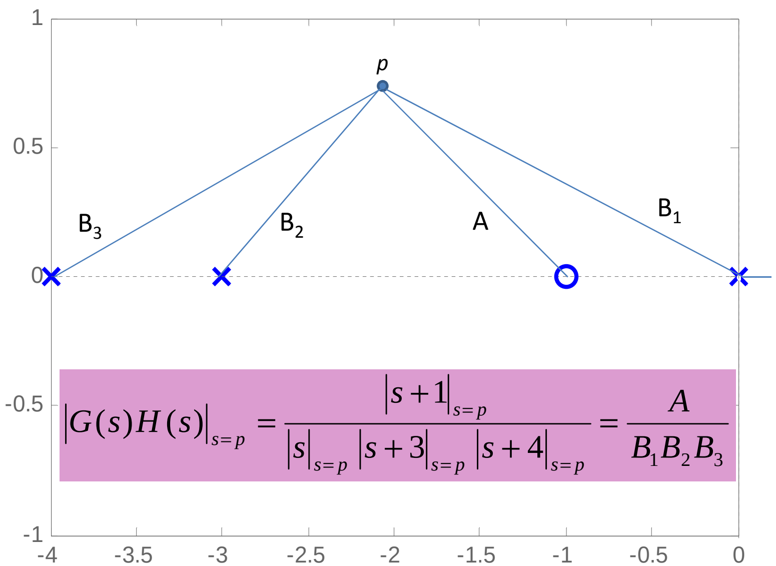
Angle and Magnitude Conditions (Graphically)



$$\angle G(s)H(s)\big|_{s=p} = \phi_1 - \theta_1 - \theta_2 - \theta_3$$

- **If angle of G(s)H(s) at *s=p* is equal to ±180°(2k+1) the point p is on root locus.**

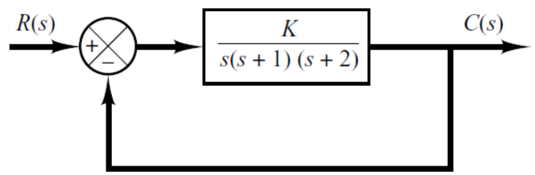
Angle and Magnitude Conditions graphically



$$\left|G(s)H(s)\right|_{s=p} = \frac{\left|s+1\right|_{s=p}}{\left|s\right|_{s=p}\left|s+3\right|_{s=p}\left|s+4\right|_{s=p}} = \frac{A}{B_1 B_2 B_3}$$

Illustrative Example -1

- Apply angle and magnitude conditions (Analytically as well as graphically) on following unity feedback system.



Illustrative Example -1

- Here $$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$
- For the given system the angle condition becomes

$$\angle G(s)H(s) = \angle \frac{K}{s(s+1)(s+2)}$$

$$\angle G(s)H(s) = \angle K - \angle s - \angle(s+1) - \angle(s+2)$$

$$\angle K - \angle s - \angle(s+1) - \angle(s+2) = \pm 180°(2k+1)$$

Illustrative Example -1

- For example to check whether ***s=-0.25*** is on the root locus or not we can apply angle condition as follows.

$$\angle G(s)H(s)\big|_{s=-0.25} = \angle K\big|_{s=-0.25} - \angle s\big|_{s=-0.25} - \angle(s+1)\big|_{s=-0.25} - \angle(s+2)\big|_{s=-0.25}$$

$$\angle G(s)H(s)\big|_{s=-0.25} = -\angle(-0.25) - \angle(0.75) - \angle(1.75)$$

$$\angle G(s)H(s)\big|_{s=-0.25} = -180° - 0° - 0°$$

$$\angle G(s)H(s)\big|_{s=-0.25} = \pm 180°(2k+1)$$

Illustrative Example -1

- Here $$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$
- And the Magnitude condition becomes

$$|G(s)H(s)| = \left|\frac{K}{s(s+1)(s+2)}\right| = 1$$

Illustrative Example -1

- Now we know from angle condition that the point **s=-0.25** is on the rot locus. But we do not know the value of gain **K** at that specific point.
- We can use magnitude condition to determine the value of gain at any point on the root locus.

$$\left|\frac{K}{s(s+1)(s+2)}\right|_{s=-0.25} = 1$$

$$\left|\frac{K}{(-0.25)(-0.25+1)(-0.25+2)}\right|_{s=-0.25} = 1$$

Illustrative Example -1

$$\left|\frac{K}{(-0.25)(-0.25+1)(-0.25+2)}\right|_{s=-0.25}=1$$

$$\left|\frac{K}{(-0.25)(0.75)(1.75)}\right|=1$$

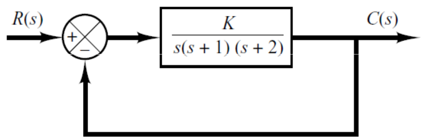
$$\left|\frac{K}{-0.3285}\right|=1$$

$$\frac{K}{0.328}=1$$

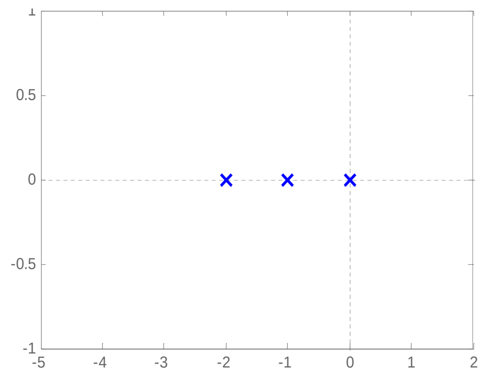
$$K=0.328$$

Construction of root loci

- **Step-1**: The first step in constructing a root-locus plot is to locate the open-loop poles and zeros in s-plane.



$$G(s)H(s)=\frac{K}{s(s+1)(s+2)}$$



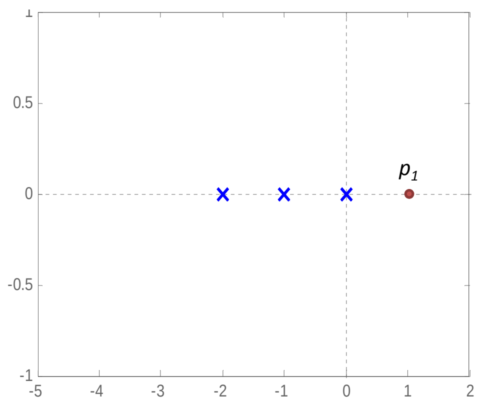
Construction of root loci

- **Step-2**: Determine the root loci on the real axis.

- To determine the root loci on real axis we select some test points.
- e.g: ***p_1*** (on positive real axis).

$$\angle s = \angle s+1 = \angle s+2 = 0^\circ$$

- The angle condition is **not satisfied.**
- Hence, there is no root locus on the positive real axis.



Construction of root loci

- **Step-2**: Determine the root loci on the real axis.

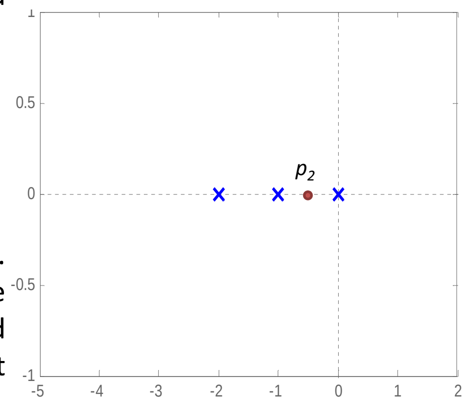
- Next, select a test point on the negative real axis between **0** and **–1**.
- Then

$$\angle s = 180^\circ, \qquad \angle s+1 = \angle s+2 = 0^\circ$$

- Thus

$$-\angle s - \angle s+1 - \angle s+2 = -180^\circ$$

- The angle condition is satisfied. Therefore, the portion of the negative real axis between **0** and **–1** forms a portion of the root locus.



Construction of root loci

- **Step-2**: Determine the root loci on the real axis.

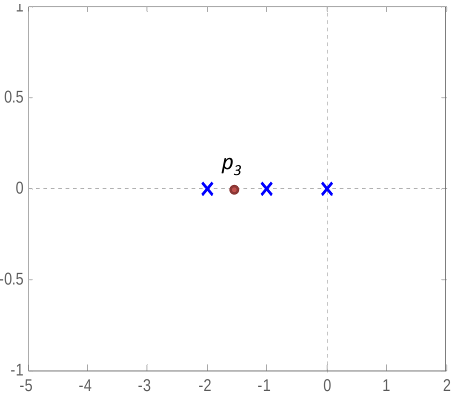
- Now, select a test point on the negative real axis between **-1** and **–2**.
- Then

$$\angle s = \angle s+1 = 180°, \qquad \angle s+2 = 0°$$

- Thus

$$-\angle s - \angle s+1 - \angle s+2 = -360°$$

- The angle condition is not satisfied. Therefore, the negative real axis between **-1** and **–2** is not a part of the root locus.

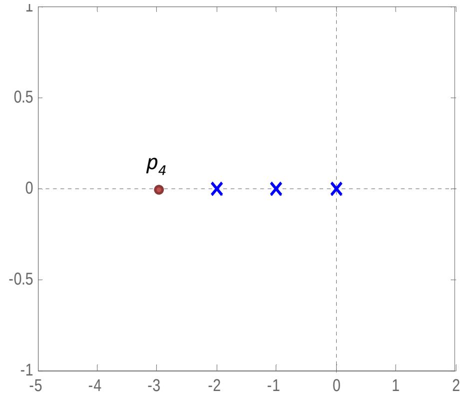


Construction of root loci

- **Step-2**: Determine the root loci on the real axis.

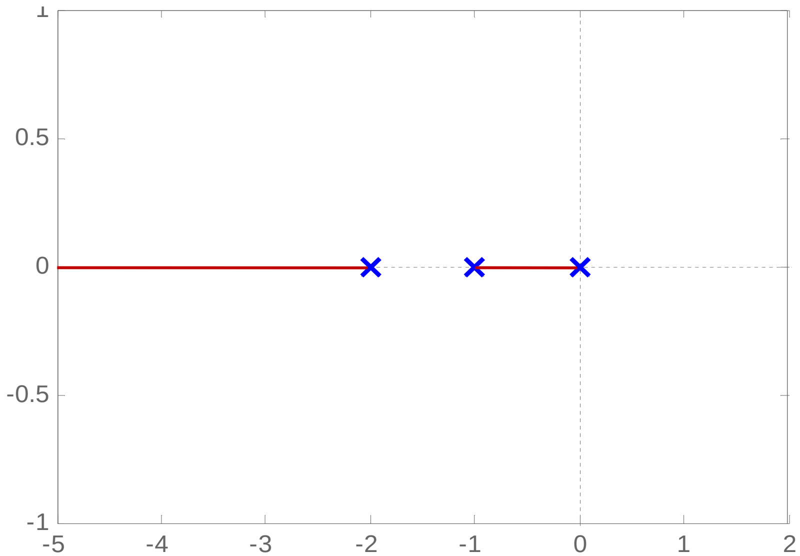
- Similarly, test point on the negative real axis between **-3** and **– ∞** satisfies the angle condition.

- Therefore, the negative real axis between **-3** and **– ∞** is part of the root locus.



Construction of root loci

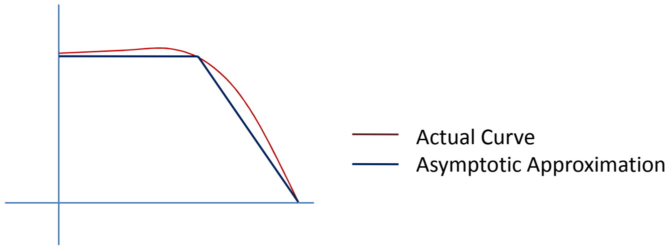
- **Step-2**: Determine the root loci on the real axis.



Construction of root loci

- **Step-3**: Determine the *asymptotes* of the root loci.

Asymptote is the straight line approximation of a curve



Construction of root loci

- Step-3: Determine the *asymptotes* of the root loci.

$$Angle\ of\ \ asymptotes = \theta = \frac{\pm 180°(2k+1)}{n-m}$$

- where
- n-----> number of poles
- m-----> number of zeros
- For this Transfer Function $G(s)H(s) = \dfrac{K}{s(s+1)(s+2)}$

$$\theta = \frac{\pm 180°(2k+1)}{3-0}$$

Construction of root loci

- Step-3: Determine the *asymptotes* of the root loci.

$$\begin{aligned} \theta &= \pm 60° && \text{when } k=0 \\ &= \pm 180° && \text{when } k=1 \\ &= \pm 300° && \text{when } k=2 \\ &= \pm 420° && \text{when } k=3 \end{aligned}$$

- Since the angle repeats itself as k is varied, the distinct angles for the asymptotes are determined as 60°, –60°, -180°and 180°.
- Thus, there are three asymptotes having angles 60°, –60°, 180°.

Construction of root loci

- Step-3: Determine the *asymptotes* of the root loci.
- Before we can draw these asymptotes in the complex plane, we must find the point where they intersect the real axis.
- Point of intersection of asymptotes on real axis (or centroid of asymptotes) can be find as out

$$\sigma = \frac{\sum poles - \sum zeros}{n-m}$$

Construction of root loci

- Step-3: Determine the *asymptotes* of the root loci.
- For $G(s)H(s) = \dfrac{K}{s(s+1)(s+2)}$

$$\sigma = \frac{(0-1-2)-0}{3-0}$$

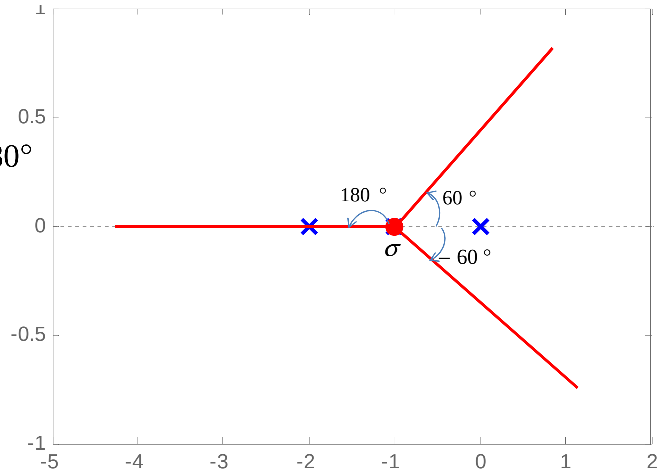
$$\sigma = \frac{-3}{3} = -1$$

Construction of root loci

- Step-3: Determine the *asymptotes* of the root loci.

$$\theta = 60^\circ , -60^\circ , 180^\circ$$

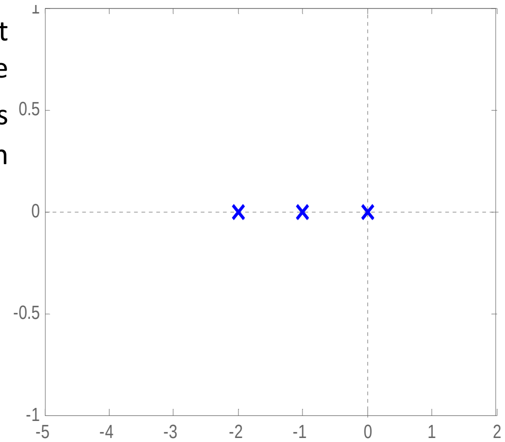
$$\sigma = -1$$



Construction of root loci

- Step-4: Determine the *breakaway point*.

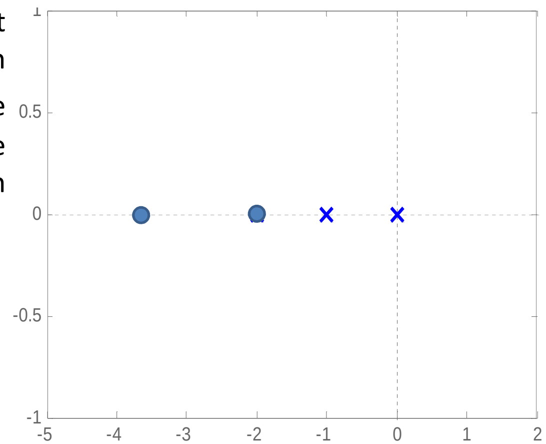
  - The breakaway point corresponds to a point in the **s** plane where multiple roots of the characteristic equation occur.



Construction of root loci

- Step-4: Determine the *break-in point*.

  - The break-in point corresponds to a point in the **s** plane where multiple roots of the characteristic equation occur.



Construction of root loci

- Step-4: Determine the *breakaway point* or *break-in point*.

  - The breakaway or break-in points can be determined from the roots of

$$\frac{dK}{ds} = 0$$

  - It should be noted that not all the solutions of **dK/ds=0** correspond to actual breakaway points.

  - If a point at which **dK/ds=0** is on a root locus, it is an actual breakaway or break-in point.

  - Stated differently, if at a point at which **dK/ds=0** the value of K takes a real positive value, then that point is an actual breakaway or break-in point.

Construction of root loci

- Step-4: Determine the *breakaway point* or *break-in point*.

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

- The characteristic equation of the system is

$$1 + G(s)H(s) = 1 + \frac{K}{s(s+1)(s+2)} = 0$$

$$\frac{K}{s(s+1)(s+2)} = -1$$

$$K = -[s(s+1)(s+2)]$$

- The breakaway point can now be determined as

$$\frac{dK}{ds} = -\frac{d}{ds}[s(s+1)(s+2)]$$

Construction of root loci

- Step-4: Determine the *breakaway point* or *break-in point*.

$$\frac{dK}{ds} = -\frac{d}{ds}[s(s+1)(s+2)]$$

$$\frac{dK}{ds} = -\frac{d}{ds}[s^3 + 3s^2 + 2s]$$

$$\frac{dK}{ds} = -3s^2 - 6s - 2$$

- Set *dK/ds=0* in order to determine breakaway point.

$$-3s^2 - 6s - 2 = 0$$

$$3s^2 + 6s + 2 = 0$$

$$s = -0.4226$$
$$= -1.5774$$

Construction of root loci

- Step-4: Determine the *breakaway point* or *break-in point*.

$$s = -0.4226$$
$$= -1.5774$$

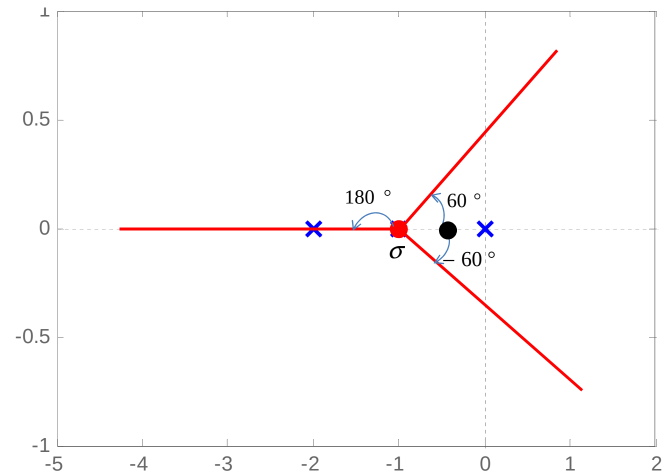
- Since the breakaway point must lie on a root locus between 0 and –1, it is clear that *s=–0.4226* corresponds to the actual breakaway point.
- Point s=–1.5774 is not on the root locus. Hence, this point is not an actual breakaway or break-in point.
- In fact, evaluation of the values of K corresponding to s=–0.4226 and s=–1.5774 yields

$$K = 0.3849, \quad \text{for } s = -0.4226$$
$$K = -0.3849, \quad \text{for } s = -1.5774$$

Construction of root loci

- Step-4: Determine the *breakaway point*.

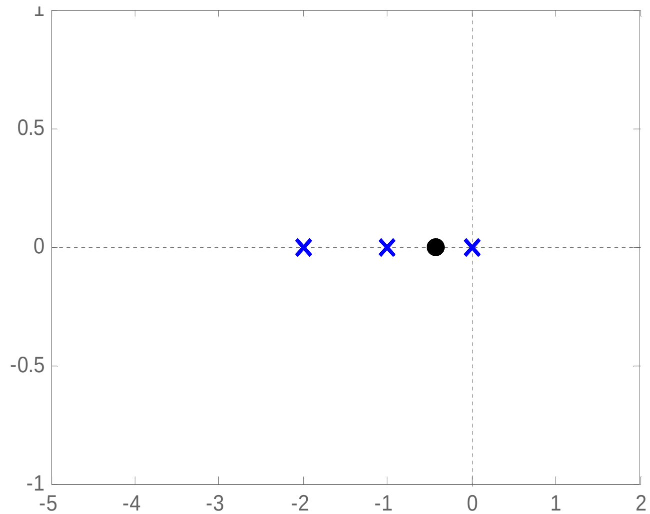
$$s = -0.4226$$



Construction of root loci

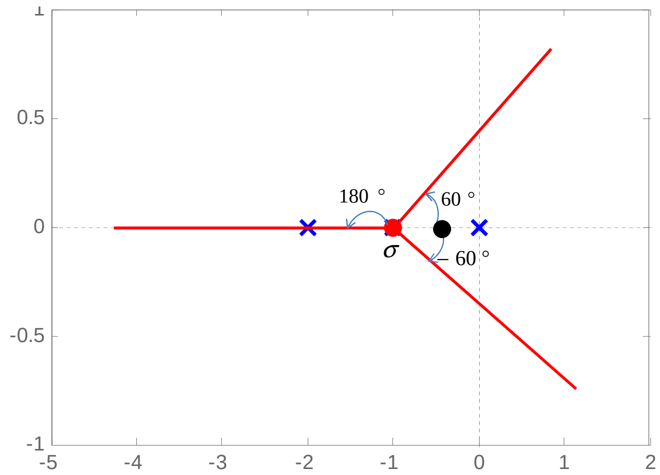
- Step-4: Determine the *breakaway point*.

$s = -0.4226$



Construction of root loci

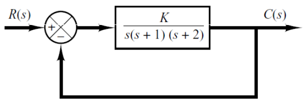
- Step-5: Determine the points where root loci cross the imaginary axis.



Construction of root loci

- Step-5: Determine the points where root loci cross the imaginary axis.
  - These points can be found by use of Routh's stability criterion.
  - Since the characteristic equation for the present system is

$$s^3 + 3s^2 + 2s + K = 0$$



  - The Routh Array Becomes

s^3	1	2
s^2	3	K
s^1	$\frac{6-K}{3}$	
s^0	K	

Construction of root loci

- Step-5: Determine the points where root loci cross the imaginary axis.

- The value(s) of K that makes the system marginally stable is 6.
- The crossing points on the imaginary axis can then be found by solving the auxiliary equation obtained from the s^2 row, that is,

$$3s^2 + K = 3s^2 + 6 = 0$$

s^3	1	2
s^2	3	K
s^1	$\frac{6-K}{3}$	
s^0	K	

- Which yields

$$s = \pm j\sqrt{2}$$

Construction of root loci

- Step-5: Determine the points where root loci cross the imaginary axis.

- An **alternative approach** is to let s=jω in the characteristic equation, equate both the real part and the imaginary part to zero, and then solve for ω and K.

- For present system the characteristic equation is

$$s^3 + 3s^2 + 2s + K = 0$$

$$(j\omega)^3 + 3(j\omega)^2 + 2j\omega + K = 0$$

$$(K - 3\omega^2) + j(2\omega - \omega^3) = 0$$

Construction of root loci

- Step-5: Determine the points where root loci cross the imaginary axis.

$$(K - 3\omega^2) + j(2\omega - \omega^3) = 0$$

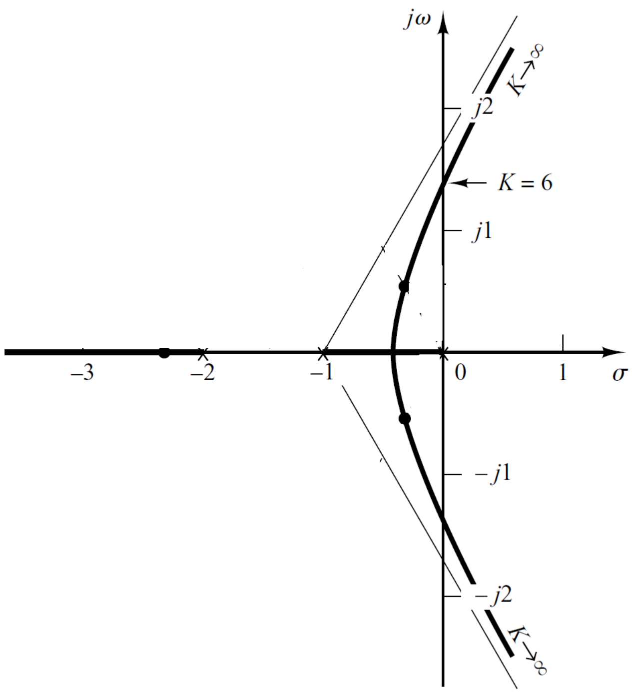
- Equating both real and imaginary parts of this equation to zero

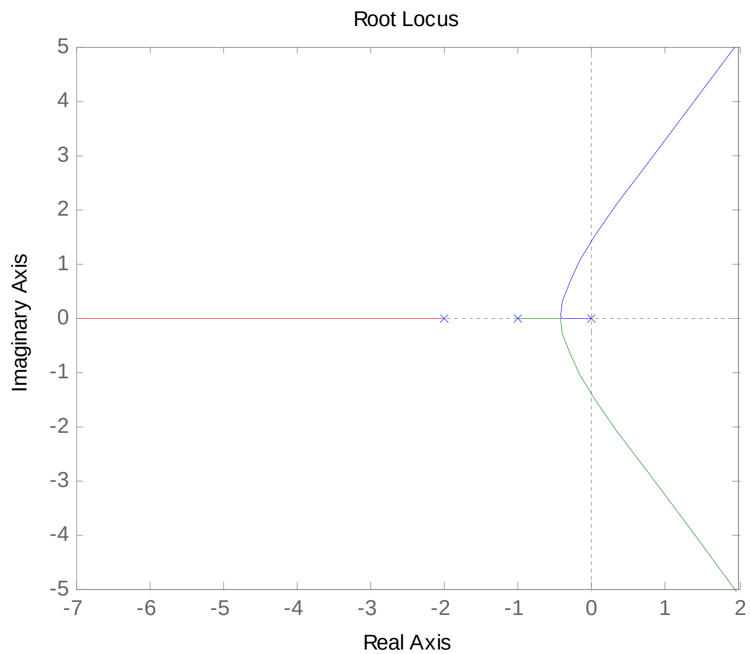
$$(2\omega - \omega^3) = 0$$

$$(K - 3\omega^2) = 0$$

- Which yields

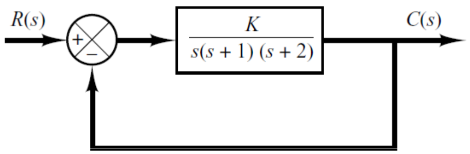
$$\omega = \pm\sqrt{2}, \quad K = 6 \quad \text{or} \quad \omega = 0, \quad K = 0$$





Example -1

- Consider following unity feedback system.



- Determine the value of **K** such that the **damping ratio** of a pair of dominant complex-conjugate closed-loop poles is **0.5**.

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

Example -1

- The damping ratio of 0.5 corresponds to

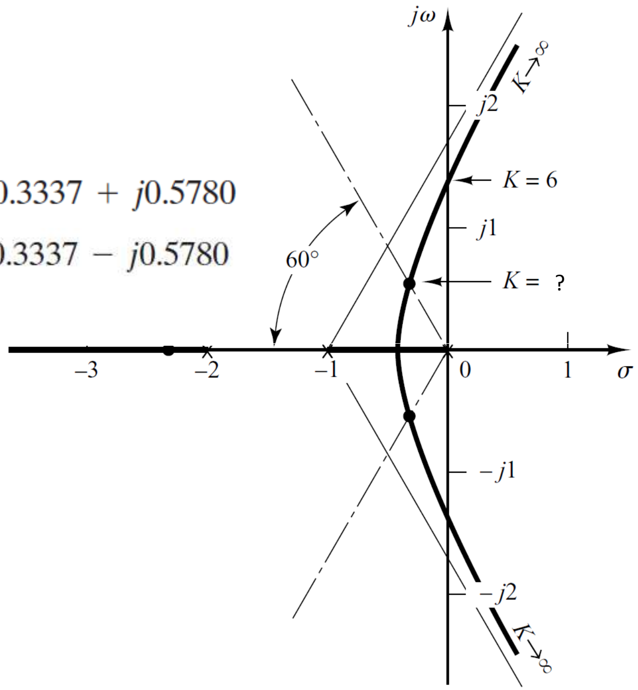
$$\zeta = \cos\beta$$

$$\beta = \cos^{-1}\zeta$$

$$\beta = \cos^{-1}(0.5) = 60^\circ$$

$$s_1 = -0.3337 + j0.5780$$

$$s_2 = -0.3337 - j0.5780$$

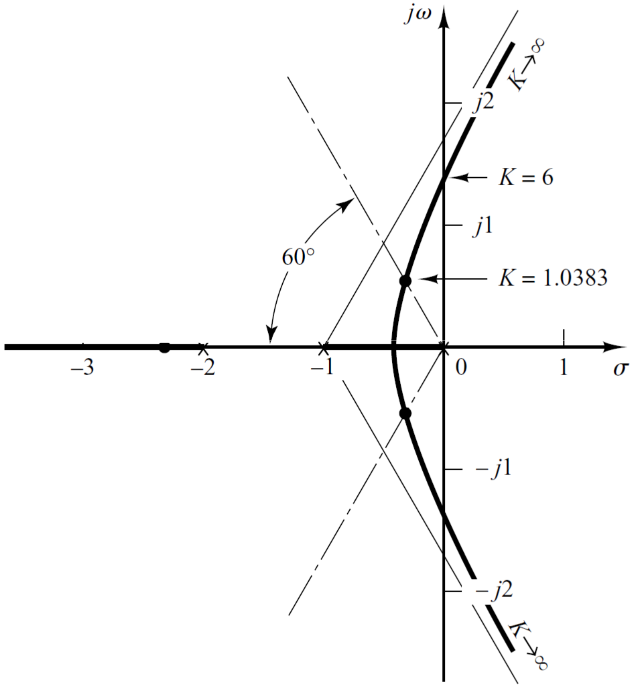


Example -1

- The value of K that yields such poles is found from the magnitude condition

$$\left|\frac{K}{s(s+1)(s+2)}\right|_{s=-0.3337+j0.5780} = 1$$

$$K = |s(s+1)(s+2)|_{s=-0.3337+j0.5780}$$
$$= 1.0383$$



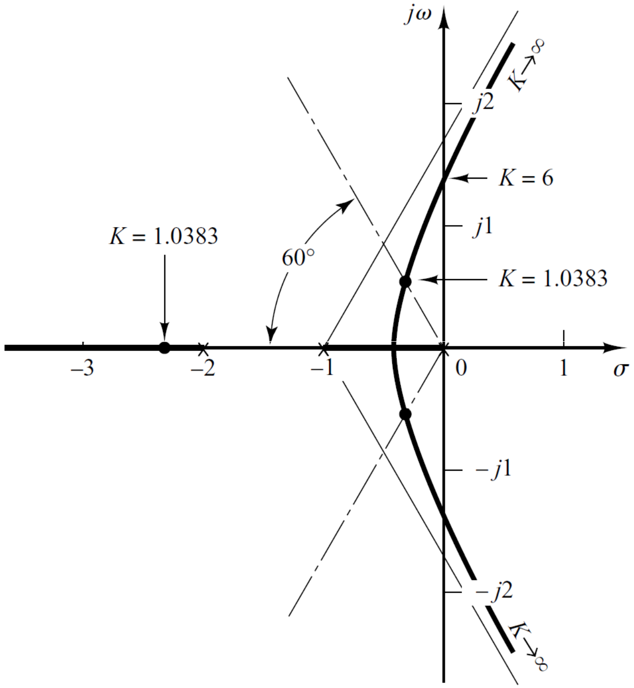
Example -1

- The third closed loop pole at K=1.0383 can be obtained as

$$1+G(s)H(s)=1+\frac{K}{s(s+1)(s+2)}=0$$

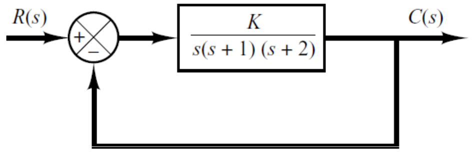
$$1+\frac{1.0383}{s(s+1)(s+2)}=0$$

$$s(s+1)(s+2)+1.0383=0$$



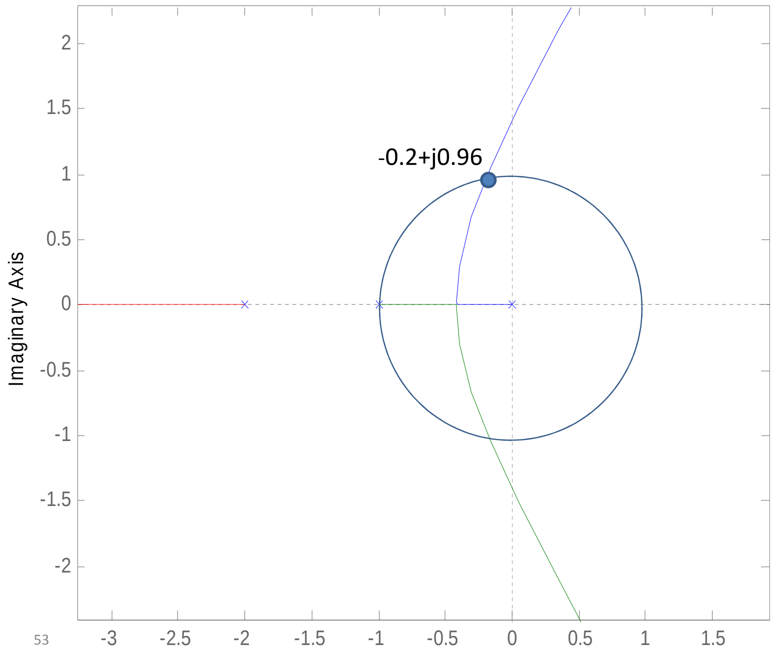
Home Work

- Consider following unity feedback system.



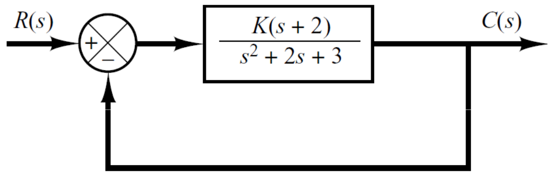
- Determine the value of **K** such that the natural undamped frequency of dominant complex-conjugate closed-loop poles is 1 rad/sec.

$$G(s)H(s)=\frac{K}{s(s+1)(s+2)}$$



Example

- Sketch the root-locus plot of following system with complex-conjugate open loop poles.



$$G(s) = \frac{K(s+2)}{s^2+2s+3}, \qquad H(s) = 1$$

$G(s)$ has a pair of complex-conjugate poles at

$$s = -1 + j\sqrt{2}, \quad s = -1 - j\sqrt{2}$$

Example

- ***Step-1:*** Pole-Zero Mao

- ***Step-2:*** Determine the root loci on real axis

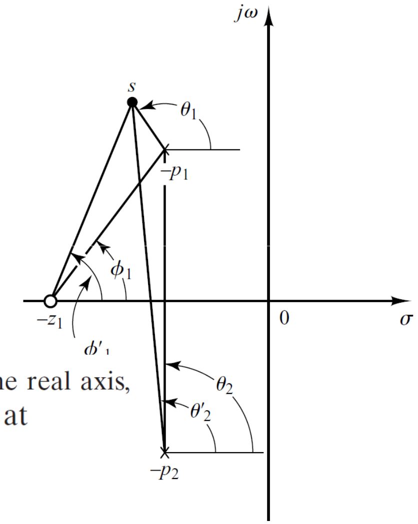
- ***Step-3***: Asymptotes

Example

- ***Step-4:*** Determine the angle of departure from the complex-conjugate open-loop poles.
  - The presence of a pair of complex-conjugate open-loop poles requires the determination of the angle of departure from these poles.
  - Knowledge of this angle is important, since the root locus near a complex pole yields information as to whether the locus originating from the complex pole migrates toward the real axis or extends toward the asymptote.

Example

- ***Step-4:*** Determine the angle of departure from the complex-conjugate open-loop poles.



The angle of departure is then

$$\theta_1 = 180° - \theta_2 + \phi_1$$
$$= 180° - 90° + 55° = 145°$$

Since the root locus is symmetric about the real axis, the angle of departure from the pole at

$$s = -p_2 \text{ is } -145°$$

Example

- ***Step-5:*** Break-in point

$$K = -\frac{s^2 + 2s + 3}{s + 2}$$

$$\frac{dK}{ds} = -\frac{(2s + 2)(s + 2) - (s^2 + 2s + 3)}{(s + 2)^2} = 0$$

$$s^2 + 4s + 1 = 0$$

$$s = -3.7320 \qquad \text{or} \qquad s = -0.2680$$